Response on the Geopotential and Geopotential Surface

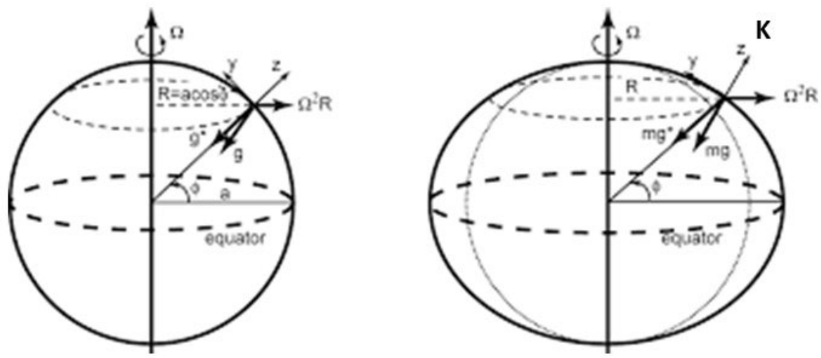
"*I have read the manuscript by Peter Chu, and while I found it quite thought-provoking, I am forced to conclude that it is actually quite misleading and do not recommend publication in its present form. It seems to me that the mistake the author is making is to formulate the equations of motion in spherical coordinates from the beginning. This is not my understanding of how the equations of motion used by models of the atmosphere and ocean are formulated. Rather, these models use a coordinate system in which the vertical direction is defined as being perpendicular to geopotential surfaces so that gravity always points along the vertical direction with no horizontal component*."

The geopotential and geopotential surface used in oceanography and meteorology are the normal geopotential and normal geopotential surface, but not the TRUE GEOPOTENTIAL and TRUE GEOPOTENTIAL SURFACE.

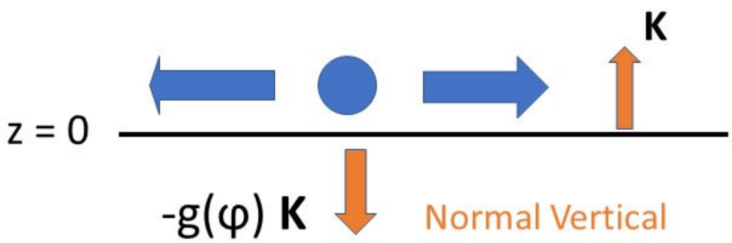
The two attached figures illustrate the difference between the normal gravity which is called the effective gravity and used in oceanography and meteorology, and the true gravity which is the most important variable in geodesy.

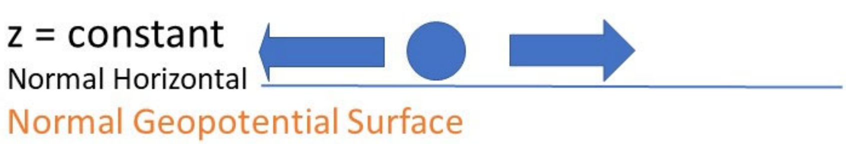
Figure A shows the main features of the effective gravity [$-g(\varphi)\mathbf{K}$]: (1) it is determined from the solid Earth with rotation and uniform mass density; (2) the unit vector **K** is perpendicular to the z surface (z = constant) and points the normal vertical; (3) the z surface is the normal horizontal and coincides with the normal geopotential surface; (4) any movement on the z surface (i.e., normal geopotential surface) is not against the normal gravity.

(A) Normal (or called Effective) Gravity
(Uniform Mass Density inside the Solid Earth)
Normal Geopotential Surface = z Surface



The two figures are from the website:
https://atoc.colorado.edu/~cassano/atoc5050/Lecture_Notes/hh_ch1.pdf





Any movement on z = constant (i.e., z surfaces) is not against the effective gravity.
The normal geopotential surface coincides with the z-surface.

Fig. A. Illustration of normal gravity, normal geopotential, normal vertical, and normal horizontal, which are used in atmospheric and oceanic dynamics.

Figure B shows the main features of the true gravity [$\mathbf{g}(\lambda, \varphi, z) = -g(\varphi)\mathbf{K} + \delta\mathbf{g}$]: (1) it is determined from the solid Earth with rotation and non-uniform mass density; (2) the true gravity has never been used in oceanography and meteorology; (3) the true gravity vector $\mathbf{g}(\lambda, \varphi, z)$ is perpendicular to the true geopotential surface such as the geoid surface, which represents the true horizontal; (4) any movement on the true geopotential surface is not against the true gravity; (5) any movement on the z-surface is against the true gravity. An additional force, the gravity disturbance (T), shows up in the z-surface momentum equations, such as in Equation (18) of the manuscript.

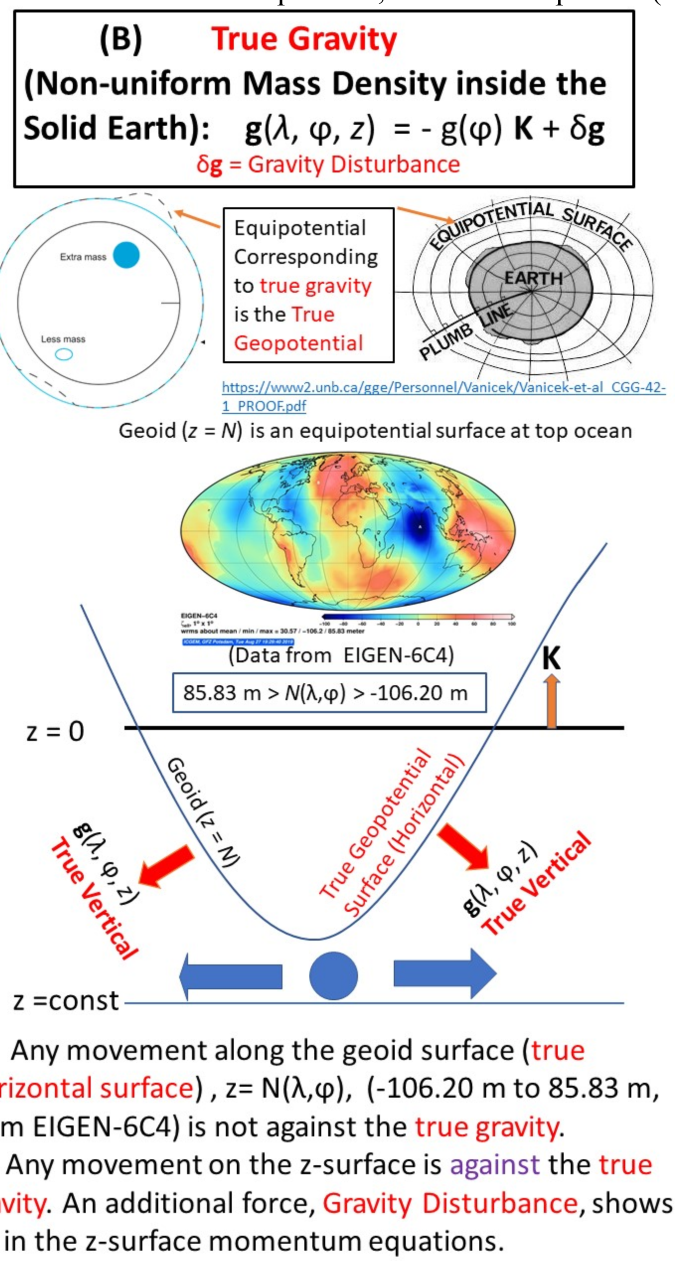


Fig. B. Illustration of true gravity, true geopotential, true vertical, and true horizontal, which should be used in atmospheric and oceanic dynamics.

Response to the Coordinate System

The ocean dynamics to include the effect of the gravity disturbance shouldn't be depend on coordinate system. I use the vector form to redrive the combined Sverdrup-Stommel-Munk equation starting from Equation (12) in the manuscript.

2 Basic equations

Large-scale ocean circulation under the Boussinesq approximation is governed by the momentum equation (Chu, 2021a)

$$\rho_0\left[\frac{D\mathbf{U}}{Dt}+f\mathbf{k}\times\mathbf{U}\right]=-\nabla_3 p+\rho\mathbf{g}+\rho_0(\mathbf{F}_h+\mathbf{F}_v) \tag{12}$$

and the continuity equation

$$\nabla\bullet\mathbf{U}+\frac{\partial w}{\partial z}=0 \tag{13}$$

where $\nabla_3\equiv\nabla+\mathbf{k}\partial/\partial z$, is the three-dimensional vector differential operator; D/Dt is total time derivative; $\mathbf{U}=(u, v)$, is the 2D velocity vector in $(\mathbf{i}, \mathbf{j})$ surfaces; w is the z-component velocity; p is the pressure; ρ is the sea water density; ρ_0 = 1,028 kg m^{-3}, is the reference density; $f = 2\Omega \sin\varphi$, is the Coriolis parameter with $\Omega = 2\pi/(86164\ \text{s})$; $(\mathbf{F}_h, \mathbf{F}_v)$ are the frictional forces with lateral and z-directional shears represented by

$$\mathbf{F}_h=A\nabla^2\mathbf{U},\quad \mathbf{F}_v=\frac{\partial}{\partial z}\left(K\frac{\partial\mathbf{U}}{\partial z}\right) \tag{14}$$

where (A, K) are the corresponding eddy viscosities.

With the constant reference density ρ_0, we can define a reference hydrostatic pressure p_0 that exactly balance the component of the true gravity $\mathbf{g}$ [$=\nabla_3\Phi,\ \ \Phi=g_0z-T\approx g_0z-g_0N(\lambda,\varphi)$] associated with reference density; i.e.,

$$-\nabla_3 p_0-\rho_0\nabla_3\Phi=0 \tag{15}$$

where

$$p_0=-\rho_0g_0z+\rho_0T\approx-\rho_0g_0(z-N) \tag{16}$$

is the hydrostatic pressure. Subtraction of (15) from (12) leads to

$$\rho_0\left[\frac{D\mathbf{U}}{Dt}+f\mathbf{k}\times\mathbf{U}\right]=-\nabla\hat{p}+(\rho-\rho_0)\nabla T+\rho_0(\mathbf{F}_h+\mathbf{F}_v) \tag{17a}$$

$$\frac{\partial\hat{p}}{\partial z}=-(\rho-\rho_0)g_0 \tag{17b}$$

where $\hat{p}=p-p_0$, is the dynamic pressure.

3 Combined Sverdrup-Stommel-Munk equation

For steady-state low Rossby number (negligible nonlinear advection) flow with friction (i.e., $D\mathbf{U}/Dt = 0$, and $\mathbf{F}_h \neq 0$, $\mathbf{F}_v \neq 0$), Eq.(17a) is simplified into

$$\rho_0 \left[f\mathbf{k} \times \mathbf{U} - A\nabla^2\mathbf{U} - \frac{\partial}{\partial z}\left(K \frac{\partial \mathbf{U}}{\partial z} \right) \right] = -\nabla \hat{p} + \left(\rho - \rho_0 \right) \nabla T \tag{18}$$

where (14) is used for $\mathbf{F}_h$ and $\mathbf{F}_v$. The turbulent momentum flux is given by

$$\rho_0 K \frac{\partial \mathbf{U}}{\partial z} |_{z=0} = \boldsymbol{\tau} \tag{19a}$$

at the rigid-lid ocean surface ($z = 0$) with $\boldsymbol{\tau}$ the wind stress, and given by

$$K \frac{\partial \mathbf{U}}{\partial z} |_{z=-H} = \gamma \mathbf{M}, \quad \mathbf{M} \equiv \int_{-H}^{0} \mathbf{U} dz \tag{19b}$$

at the lower boundary ($z = -H$). Here, $\mathbf{M}$ is the volume transport; γ is the Rayleigh friction coefficient (Stommel 1948). When $\gamma = 0$, Eq.(19b) represents negligible turbulent momentum flux at $z = -H$ (Sverdrup 1947, Munk 1950). Integration of (18) in the z-direction from $z = -H$ to $z = 0$ and use of (19a) and (19b) leads to

$$\left[f\mathbf{k} \times \mathbf{M} - A\nabla^2\mathbf{M} - \left(\boldsymbol{\tau} - \gamma \mathbf{M} \right) \right] = -\int_{-H}^{0} \nabla \hat{p} dz + \int_{-H}^{0} \left[\left(\rho - \rho_0 \right) \nabla T \right] dz \tag{20}$$

Curl of the vector equation (20) gives

$$\nabla \times \left[f\mathbf{k} \times \mathbf{M} - A\nabla^2\mathbf{M} - \left(\boldsymbol{\tau} - \gamma \mathbf{M} \right) \right] = \int_{-H}^{0} \left[\nabla \rho \times \nabla T \right] dz \tag{21}$$

Let the volume transport stream-function (Ψ) be defined by

$$\nabla \Psi = -\frac{1}{\rho_0} \mathbf{k} \times \mathbf{M}, \qquad \text{i.e., } \mathbf{M} = \rho_0 \mathbf{k} \times \nabla \Psi \tag{22}$$

Substitution of (22) into (21) leads to

$$\nabla \times \left[-f \nabla \Psi - A\nabla^2 \left(\mathbf{k} \times \nabla \Psi \right) + \gamma \left(\mathbf{k} \times \nabla \Psi \right) \right] = \frac{1}{\rho_0} \left[\nabla \times \boldsymbol{\tau} + \int_{-H}^{0} \left(\nabla \rho \times \nabla T \right) dz \right] \tag{23}$$

Since

$$\nabla \times \left(\mathbf{k} \times \nabla \Psi \right) = \mathbf{k} \nabla^2 \Psi, \ \nabla \times (-f \nabla \Psi) = \beta \frac{\partial \Psi}{\partial x}, \quad \beta \equiv \frac{df}{dy} \quad (\beta \text{ coefficient}) \tag{24}$$

substituting (24) into (23) and conducting inner product with the unit vector **k**, we obtain the combined Sverdrup-Stommel-Munk equation

$$\boxed{-A\nabla^4 \Psi + \gamma \nabla^2 \Psi + \beta \frac{\partial \Psi}{\partial x} = \frac{1}{\rho_0} \left[\text{curl } \boldsymbol{\tau} + \int_{-H}^{0} \mathbf{k} \bullet \left(\nabla \rho \times \nabla T \right) dz \right]} \tag{25}$$

with an additional gravity disturbance forcing (GDF),

$$\text{GDF} = \int_{-H}^{0} \mathbf{k} \bullet \left(\nabla \rho \times \nabla T \right) dz = \int_{-H}^{0} J(\rho, T) dz, \quad J(\rho, T) \equiv \frac{\partial \rho}{\partial x} \frac{\partial T}{\partial y} - \frac{\partial T}{\partial x} \frac{\partial \rho}{\partial y} \tag{26}$$

where $J(\rho, T)$ is the Jacobian of ρ and T.