Choosing a scalar product to make the CSW propagation operator anti-self-adjoint.

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Abstract

My issue with the use of C-EOFs in such a problem is that the correlation matrix that is diagonalized is by construction Hermitian. As is well know, this entails that its eigenmodes are *always* orthogonal for the euclidean scalar product, whereas the standard (Huthnance 1975) paper states very clearly that CSW modes are *not* orthogonal for this scalar product (and there is little hope the introduction of stratification should make things any easier in this respect). In my opinion, this means that considering the successive EOF eigenmodes (orthogonal by construction) as in one-to-one correspondance with the CSW modes (classically not orthogonal) may happen to be justifiable by a careful analysis, but is definitely not a trivial result... In this text I show that a scalar product can be found for which the different CSW modes are orthogonal. This scalar product involves the sea surface elevation perturbation, which does not seem available to (Brunner and Lwiza 2020, subm.).

1. Basic linear algebra results.

1a. *For matrices*

The argument is very classic, I reproduce it here for the sake of completeness. Let a T superscript denote the transpose of a matrix, an asterisk the element-wise complex conjugate of a matrix, and an H superscript the Hermitian conjugate of a matrix, obtained by transposing its element-wise complex conjugate. Let $\mathbf{M}$ be a Hermitian matrix, *i.e.* such that $\mathbf{M}^H = (\mathbf{M}^T)^* = \mathbf{M}$. Let $\mathbf{X}$ and $\mathbf{Y}$ be two of its eigenvectors, associated respectively to the eigenvalues λ and μ.

Then a first series of manipulations shows:

$$\mathbf{X}^H\mathbf{M}\mathbf{X} = \lambda\mathbf{X}^H\mathbf{X} = \lambda||\mathbf{X}||^2.$$

Then

$$(\mathbf{X}^H\mathbf{M}\mathbf{X})^H = \lambda^*||\mathbf{X}||^2,$$

but in the meantime

$$(\mathbf{X}^H\mathbf{M}\mathbf{X})^H = \mathbf{X}^H\mathbf{M}^H\mathbf{X} = \mathbf{X}^H\mathbf{M}\mathbf{X} = \lambda||\mathbf{X}||^2$$

hence:

$$\lambda^*||\mathbf{X}||^2 = \lambda||\mathbf{X}||^2.$$

Any eigenvalue of an hermitian matrix is thus real.

In a second stage, one sees that:

$$\mathbf{X}^H(\mathbf{M}\mathbf{Y}) = (\mathbf{M}\mathbf{X})^H\mathbf{Y}$$

entails

$$\mu\mathbf{X}^H\mathbf{Y} = \lambda^*\mathbf{X}^H\mathbf{Y}$$

hence, as $\lambda^* = \lambda$:

$$[\mu - \lambda]\, \mathbf{X}^H \mathbf{Y} = 0.$$

Any two eigenvectors of an hermitian matrix are thus orthogonal, or associated to the same eigenvalue.

This last result carries over to the case of an anti-Hermitian matrix (*i.e.* such that $\mathbf{M}^H = -\mathbf{M}$), though one can show that any eigenvalue λ is in this case pure imaginary.

As the matrix that is analyzed in the C-EOF method is by construction Hermitian, the EOF modes are *always* orthogonal, and the method is not adapted to the analysis of a system in which the main modes of motion have no reason to (or have good reasons not to) be orthogonal.

1b. *For linear operators*

The considerations above carry quite straightforwardly over from the matrix setting to the linear differential operators setting. In this case, one considers a scalar product, denoted as $\langle \cdot ; \cdot \rangle$ and a linear operator $\mathcal{L}$ operating on a functional space. Then one considers the ajoint linear operator, $\mathcal{L}^+$, such that for all fields x and y in suitable functional spaces:

$$\left\langle \mathcal{L}^+ x; y \right\rangle = \langle x; \mathcal{L} y \rangle$$

Then there is a close correspondance between the properties of an Hermitian matrix and those of a self-adjoint operator (*i.e.* such that $\mathcal{L}^+ = \mathcal{L}$), and between the properties of an anti-Hermitian matrix and those of an anti-self-adjoint operator (*i.e.* such that $\mathcal{L}^+ = -\mathcal{L}$).

2. Changing the scalar product

One thing one should be careful about, though, is that the adjoint of an operator depends on the scalar product used. An operator that is not self-adjoint for a given scalar product can sometimes be made self-adjoint by finding a scalar product that "suits" it better. The orthogonality property of eigenvectors associated to different eigenvalues is then satisfied for the scalar product for which the operator is self-adjoint. Since in the original (Huthnance 1975) paper the phase velocities of CSWs are said to be purely real, it seems there is hope that another choice of scalar product could make the operator either self-adjoint or anti-self-adjoint. Finding a way to insert the discretized version of this scalar product in the C-EOF matrix could allow a more convincing separation of the different CSW modes.

2a. *In the CSW problem*

In the notations of the (Huthnance 1975) paper, the equations of motion are summarized as (with D^2 a constant):

$$\partial_t \left| \begin{array}{l} u \\ v \\ \zeta \end{array} \right. = \begin{bmatrix} 0 & 1 & -\partial_x \\ -1 & 0 & -\partial_y \\ -\frac{1}{D^2}\partial_x(h\cdot) & -\frac{1}{D^2}\partial_y(h\cdot) & 0 \end{bmatrix} \left| \begin{array}{l} u \\ v \\ \zeta \end{array} \right. , \tag{1}$$

with $hu \to 0$ for $x \to 0$, $\zeta \to 0$ for $x \to \infty$ as boundary conditions.

Heuristically, I find it convenient to introduce a change of variables to bring the matrix on the right-hand-side as close as possible to anti-self-adjoint form (dividing ζ by h probably brings in complications close to the shore).

$$\partial_t \left| \begin{array}{l} u \\ v \\ \zeta \frac{D}{h} \end{array} \right. = \begin{bmatrix} 0 & 1 & -\frac{1}{D}\partial_x(h\cdot) \\ -1 & 0 & -\frac{1}{D}\partial_y(h\cdot) \\ -\frac{1}{Dh}\partial_x(h\cdot) & -\frac{1}{Dh}\partial_y(h\cdot) & 0 \end{bmatrix} \left| \begin{array}{l} u \\ v \\ \zeta \frac{D}{h} \end{array} \right. . \tag{2}$$

With this change of variables, the equations of motion can be written as:

$$\partial_t \left(\mathbf{u}, \frac{D}{h}\zeta \right) = \mathcal{L} \left(\mathbf{u}, \frac{D}{h}\zeta \right),$$

with

$$\mathcal{L} : (\mathbf{B}, g) \longrightarrow \left(-\mathbf{k} \wedge \mathbf{B} - \frac{1}{D}\mathrm{grad}(hg), -\frac{1}{Dh}\mathrm{div}(h\mathbf{B})\right).$$

Finding a scalar product for which $\mathcal{L}$ is self-adjoint amounts to finding a scalar product for which,

$$\forall (\mathbf{A}, f), (\mathbf{B}, g), \langle (\mathbf{A}, f); \mathcal{L}(\mathbf{B}, g)\rangle = \langle \mathcal{L}(\mathbf{A}, f); (\mathbf{B}, g)\rangle$$

Taking as scalar product

$$\langle (\mathbf{A}, f); (\mathbf{B}, g)\rangle = \int_{\mathcal{D}} \left[h\mathbf{A}^* \cdot \mathbf{B} + h^2 f^* g\right] dS,$$

where the integration is over the complete study domain, one can perform the following manipulations:

$$\begin{aligned}
\langle (\mathbf{A}, f); \mathcal{L}(\mathbf{B}, g)\rangle &= \int_{\mathcal{D}} \left[h\mathbf{A}^* \cdot \left[-\mathbf{k} \wedge \mathbf{B} - \frac{1}{D}\mathrm{grad}(hg)\right] + h^2 f^* \left[-\frac{1}{Dh}\mathrm{div}(h\mathbf{B})\right]\right] dS \\
&= \int_{\mathcal{D}} \left[h\mathbf{B} \cdot (\mathbf{k} \wedge \mathbf{A}^*) - \frac{1}{D} h\mathbf{A}^* \cdot \mathrm{grad}(hg) - \frac{1}{D} hf^* \mathrm{div}(h\mathbf{B})\right] dS \\
&= \int_{\mathcal{D}} \left[h\mathbf{B} \cdot (\mathbf{k} \wedge \mathbf{A}^*) + \frac{1}{D} hg\mathrm{div}(h\mathbf{A}^*) + \frac{1}{D} h\mathbf{B} \cdot \mathrm{grad}(hf^*)\right] dS \\
&\quad -\frac{1}{D}\int_{\mathcal{D}} \left[\mathrm{div}(h^2 f^* \mathbf{B}) + \mathrm{div}(h^2 \mathbf{A}^* g)\right] dS \\
&= \int_{\mathcal{D}} \left[h\mathbf{B} \cdot \left[\mathbf{k} \wedge \mathbf{A}^* + \frac{1}{D}\mathrm{grad}(hf^*)\right] + h^2 g \left[\frac{1}{Dh}\mathrm{div}(h\mathbf{A}^*)\right]\right] dS \\
&\quad -\frac{1}{D}\int_{\mathcal{D}} \left[\mathrm{div}(h^2 f^* \mathbf{B}) + \mathrm{div}(h^2 \mathbf{A}^* g)\right] dS \\
&= -\langle \mathcal{L}(\mathbf{A}, f); (\mathbf{B}, g)\rangle - \frac{1}{D}\int_{\mathcal{D}} \left[\mathrm{div}(h^2 f^* \mathbf{B}) + \mathrm{div}(h^2 \mathbf{A}^* g)\right] dS \\
&= -\langle \mathcal{L}(\mathbf{A}, f); (\mathbf{B}, g)\rangle - \frac{1}{D}\int_{\partial\mathcal{D}} \left[h^2 f^* \mathbf{B} + h^2 \mathbf{A}^* g\right] \cdot d\mathbf{n}
\end{aligned}$$

It seems reasonable (though it probably should be checked) that the boundary term in this last equation vanishes for fields satisfying the boundary conditions imposed on the flow (one probably should use periodicity in the along-shore direction, and should check that the cross-shore transport vanishes fast enough at the coast, and that the of sea surface height perturbation vanishes fast enough far from the coast. I suspect trouble arises for the Kelvin wave). If this is the case, we see that for this scalar product the operator governing the motion of the CSWs in this framework is anti-self-adjoint, and that its eigenmodes are orthogonal. Namely, if (u_n, v_n, ζ_n) and (u_m, v_m, ζ_m) are two CSW modes with different phase velocities

$$\int_{\mathcal{D}} \left[h(u_n^* u_m + v_n^* v_m) + D^2 \zeta_n^* \zeta_m\right] dS = 0$$

this is consistent with proposition (m,ii) of (Huthnance 1975) that the energy defined in this way separates gracefully in contributions from the different CSW modes.

2b. *For a matrix diagonalization problem*

The eigenmodes obtained by the C-EOF method are orthogonal with respect to the euclidean scalar product, which is a discretized version of the scalar product

$$\langle f; g\rangle = \int_D f^* g dS,$$

where the integration domain is restricted to the domain where observations are available, and a weighting proportional to the density of observations is implicitly applied. Considering the simple case where we want the eigenvectors to be orthogonal with respect to a scalar product that has a different (but still local in space) weighting function w,

$$\langle f; g \rangle = \int_D f^* g w(\mathbf{x}) dS,$$

it seems the least one can do is to multiply the empirical covariance matrix to the left and to the right by the diagonal matrix containing the square root of the weighting function at the observation point $\mathbf{W}^{1/2}$, diagonalize

$$\mathbf{P} = \mathbf{W}^{1/2}\mathbf{M}\mathbf{W}^{1/2}$$

instead of $\mathbf{M}$, then multiply the eigenvectors by the inverse of $\mathbf{W}^{1/2}$, and call *these vectors* the empirical eigenmodes.

In this setting:
– the scalar product of two vectors $\mathbf{X}$ and $\mathbf{Y}$ is equal to $\mathbf{X}^H\mathbf{W}\mathbf{Y}$.
– if $\mathbf{X}$ and $\mathbf{Y}$ are two eigenvectors of $\mathbf{P}$ associated to different eigenvalues, they are orthogonal with respect to the Euclidean scalar product, $\mathbf{X}^H\mathbf{Y}$ (which is not the scalar product we want).
– but then $\mathbf{W}^{-1/2}\mathbf{X}$ and $\mathbf{W}^{-1/2}\mathbf{Y}$ are such that:

$$(\mathbf{W}^{-1/2}\mathbf{X})^H\mathbf{W}(\mathbf{W}^{-1/2}\mathbf{Y}) = \mathbf{X}^H\mathbf{W}^{-1/2}\mathbf{W}\mathbf{W}^{-1/2}\mathbf{Y} = \mathbf{X}^H\mathbf{Y} = 0$$

This shows that by scaling the observations in an appropriate way one can obtain eigenvectors of the covariance matrix which are orthogonal with respect to a non-trivial scalar product.

3. So what?

- Maybe the empirical eigenmodes retrieved in this way could be in better agreement with the theoretical ones?
- Is there still need for higher-order eigenmodes to explain the variance, or has the "scattering" decreased in importance?

Clearly, the authors lack sea surface height observations:

- Maybe for this type of waves the sea surface elevation signature can be derived from the surface currents?
- Maybe for this type of waves it is a negligible contribution to the scalar product?

I have not given these issues any thoughts. But the impact of the weighting by the water column height is easy to test, and I'd be surprised if it was negligible.

References

Brunner, K. and K. Lwiza, 2020, subm.: Evidence of coastal trapped wave scattering using high-frequency radar data in the Mid-Atlantic Bight. *Ocean Science Discussion*.

Huthnance, J. M., 1975: On trapped waves over a continental shelf. *J. Fluid Mech.*, **69**, 689–704.