Comment from Referee#1: 1 General comments

The manuscript investigates application of a hybrid variational-ensemble approach to a limited area model with sub-mesoscale resolution. So far constraining sub-mesoscale features remains largely out of reach of contemporary ocean forecasting with the standard set of remote and in-situ observations. This study investigates sub-mesoscale forecasting in situation with dense in-situ observations in a relatively calm region. The hybrid approaches become popular in atmospheric and ocean forecasting due to increasingly clear understanding of limitations of 4D-Var systems due to lack of mechanisms to carry information forward from previous cycles. In my view, the hybrid approaches, while succeeding in adding some degree of flow dependence to the background covariance, still remain largely empirical and lack consistent formulation. As a consequence, to the best of my knowledge, there still no published experiments with small models that would convincingly demonstrate advantages of hybrid systems over much more simple and consistent EnKF systems. Below I will list some major and minor issues I see with the manuscript and give recommendation in the conclusion.

Author's response:

We thanks Dr. P Sakov for carefully reading our manuscript and the useful comments he provided. We completely agree on the major challenges we try to address that Dr. Sakov has correctly identified, as well as strengthens and weakness of the proposed approach. In particular, we agree that part of the methodology is somehow empirical, although largely adopted, and we will better specify this point in the revised version of the manuscript.

Comment from Referee: 2 Major Issues

1. Equations (6), (7) and (8).

One problem with mixing covariance matrices is that there is no good way to incorporate it in a consistent way into the optimisation problem. In particular, the claim that linear mixing (6) can be consistent with the cost function (8) is generally wrong. The framework (6-8) assumes

$$(\mathbf{x}_c+\mathbf{x}_e)^T[\alpha\boldsymbol{B}_c+(\mathbf{1}-\alpha)\boldsymbol{B}_e\,]^{-1}(\mathbf{x}_c+\mathbf{x}_e)=(\mathbf{x}_c)^T(\alpha\boldsymbol{B}_c)^{-1}\mathbf{x}_c+(\mathbf{x}_e)^T[(\mathbf{1}-\alpha)\boldsymbol{B}_e\,]^{-1}\mathbf{x}_e$$

This implies $x_c \perp x_e$, $B_c \perp B_e$; which is generally not true and difficult (or even impossible) to impose in practice. (The same applies to Eqs. 4-6 in Wang et al. 2007.) If the above is true, then the manuscript must be modified accordingly.

Author's response:

We thank Dr. Sokov for this comment, but we only partially agree. The equality written by the reviewer does require the orthogonality. On the other hand, in order to use eq.8 instead of eq.6, it is only necessary that the cost function in eq.8 reaches the minimum for the same values of $\delta\mathbf{x}$ as the one in eq.6 and the ensemble generated by the analysis using eq.8 estimates the same covariance matrix as the one using eq.6. To demonstrate it we only need to assume that in eq.8 x_c and x_e, may be perturbed independently and that **B** *contains the true background-error covariances, i.e. the background errors are well specified. These assumptions are much weaker than imposing the orthogonality on both the x_c,*

x_e and B_c, B_e. We think that the independency between x_c and x_e is a reasonable assumption since the two variables separately sample historical events and current forecast, while the suitability of **B** *implicitly relies on the quality of the ensemble and climatological error estimates. We will add the following Appendix in the manuscript with the full derivation of eq.8.*

Author's changes in manuscript:

APPENDIX

We start from the cost function:

$$J(\delta\boldsymbol{x}) = \frac{1}{2}\delta\boldsymbol{x}^{T}\mathbf{B}^{-1}\delta\boldsymbol{x} + \frac{1}{2}(\mathbf{H}\delta\boldsymbol{x} - \mathbf{d})^{T}\mathbf{R}^{-1}(\mathbf{H}\delta\boldsymbol{x} - \mathbf{d}) \quad (A.1)$$

To define our hybrid assimilation schemes we compute **B** *as a linear combination of the "static" covariance operator,* $\mathbf{B}_c$, *and the flow-dependent operator,* $\mathbf{B}_e$:

$$\mathbf{B} = \alpha\mathbf{B}_c + (1-\alpha)\mathbf{B}_e \quad (A.2)$$

where α is the relative weight. Substituting A.2 in A.1 we obtain the new hybrid cost function:

$$J(\delta\boldsymbol{x}) = \frac{1}{2}\delta\boldsymbol{x}^{T}(\alpha\mathbf{B}_c + (1-\alpha)\mathbf{B}_e)^{-1}\delta\boldsymbol{x} + \frac{1}{2}(\mathbf{H}\delta\boldsymbol{x} - \mathbf{d})^{T}\mathbf{R}^{-1}(\mathbf{H}\delta\boldsymbol{x} - \mathbf{d}) \quad (A.3)$$

We define now the increment as a weighted sum of parts corresponding to static and flow-dependent covariance matrices:

$$\delta\boldsymbol{x} = \delta\boldsymbol{x}_{\mathbf{c}} + \delta\boldsymbol{x}_{\mathbf{e}}.$$

We want to demonstrate that:

$$J(\delta\boldsymbol{x}) = \frac{1}{2}\delta\boldsymbol{x}_{\mathbf{c}}^{T}(\alpha\mathbf{B}_c)^{-1}\delta\boldsymbol{x}_{\mathbf{c}} + \frac{1}{2}\delta\boldsymbol{x}_{\mathbf{e}}^{T}((1-\alpha)\mathbf{B}_e)^{-1}\delta\boldsymbol{x}_{\mathbf{e}} + \frac{1}{2}(\mathbf{H}\delta\boldsymbol{x} - \mathbf{d})^{T}\mathbf{R}^{-1}(\mathbf{H}\delta\boldsymbol{x} - \mathbf{d}) \quad (A.4)$$

has the minimum for the same value of $\delta\boldsymbol{x}$ as A.3.

To minimize A.4, δx_c and δx_e must satisfy $\frac{\partial J(\delta x)}{\partial x_c} = 0$ and $\frac{\partial J(\delta x)}{\partial x_e} = 0$ which gives:

$$(\alpha\mathbf{B}_c)^{-1}\delta\boldsymbol{x}_c + \frac{\partial}{\partial x_c}\left(\frac{1}{2}\delta\boldsymbol{x}_{\mathbf{e}}^{T}((1-\alpha)\mathbf{B}_e)^{-1}\delta\boldsymbol{x}_{\mathbf{e}}\right) + \frac{1}{2}\frac{\partial J_o}{\partial x_c} = 0 \quad (A.5)$$

$$[(1-\alpha)\mathbf{B}_e]^{-1}\delta\boldsymbol{x}_{\mathbf{e}} + \frac{\partial}{\partial x_e}\left(\frac{1}{2}\delta\boldsymbol{x}_{\mathbf{c}}^{T}(\alpha\mathbf{B}_c)^{-1}\delta\boldsymbol{x}_{\mathbf{c}}\right) + \frac{1}{2}\frac{\partial J_o}{\partial x_e} = 0 \quad (A.6)$$

where Jo is the observational term. Assuming that δx_c and δx_e can be perturbed independently, both the second terms on the left hand side of A.5 and A.6 are null:

$$\frac{\partial}{\partial x_c}\left(\frac{1}{2}\delta\boldsymbol{x}_{\mathbf{e}}^{T}((1-\alpha)\mathbf{B}_e)^{-1}\delta\boldsymbol{x}_{\mathbf{e}}\right) = 0, \quad (A.7)$$

$$\frac{\partial}{\partial x_e}\left(\frac{1}{2}\delta\boldsymbol{x}_{\mathbf{c}}^{T}(\alpha\mathbf{B}_c)^{-1}\delta\boldsymbol{x}_{\mathbf{c}}\right) = 0 \quad (A.8)$$

and

$$\frac{\partial J_o}{\partial x} = \frac{\partial J_o}{\partial x_e} = \frac{\partial J_o}{\partial x_c} = 2\mathbf{H}^{T}\mathbf{R}^{-1}(\mathbf{H}\delta\boldsymbol{x} - \mathbf{d}) \,. \quad (A.9)$$

This is a reasonable assumption, because the two random values are sampled from different Gaussians. Although they are defined over the same space, one is sampled from historical states, and the other *from current forecasts. Premultiplying A.5 by $\alpha\boldsymbol{B}_c$ and A.6 by $(1-\alpha)\boldsymbol{B}_e$, removing the null terms, summing the two subsequent equations and applying A.9 yields:*

$$0 = (\delta\boldsymbol{x_c}+\delta\boldsymbol{x_e}) + \frac{1}{2}[\alpha\mathbf{B}_\mathrm{c} + (1-\alpha)\mathbf{B}_\mathrm{e}]\ \frac{\partial \mathrm{J_o}}{\partial \mathrm{x}} \tag{A.10}$$

Multiplying A.10 by the inverse of the hybrid covariance:

$$0 = [\alpha\mathbf{B}_\mathrm{c} + (1-\alpha)\mathbf{B}_\mathrm{e}]^{-1}(\delta\boldsymbol{x_c}+\delta\boldsymbol{x_e}) + \mathbf{H}^T\mathbf{R}^{-1}[\mathbf{H}(\delta\boldsymbol{x_c}+\delta\boldsymbol{x_e}) - \mathbf{d}] \tag{A.11}$$

This is also the minimum of A.3 that we wanted as a proof.

Furthermore, defining the background and analysis perturbations around the true state $\boldsymbol{x_t}$ as:

$\delta\boldsymbol{x_b} = \boldsymbol{x_b} - \boldsymbol{x_t}$

and

$\delta\boldsymbol{x_a} = \boldsymbol{x_a} - \boldsymbol{x_t}$,

by adding and subtracting the true state A.11 becomes:

$$0 = [\alpha\mathbf{B}_\mathrm{c} + (1-\alpha)\mathbf{B}_\mathrm{e}]^{-1}(\boldsymbol{x_a} - \boldsymbol{x_b} - \boldsymbol{x_t} + \boldsymbol{x_t}) + \mathbf{H}^T\mathbf{R}^{-1}[\mathbf{H}(\boldsymbol{x_a} - \boldsymbol{x_b} - \boldsymbol{x_t} + \boldsymbol{x_t}) - (\boldsymbol{y} - \boldsymbol{Hx_b})]$$

or:

$$0 = [\alpha\mathbf{B}_\mathrm{c} + (1-\alpha)\mathbf{B}_\mathrm{e}]^{-1}(\delta\boldsymbol{x}_\mathbf{a} - \delta\boldsymbol{x}_\mathbf{b}) + \mathbf{H}^T\mathbf{R}^{-1}[\mathbf{H}\delta\boldsymbol{x}_\mathbf{a} - (\boldsymbol{y} - \boldsymbol{Hx_t})]$$

that can be written also as:

$$\{[\alpha\mathbf{B}_\mathrm{c} + (1-\alpha)\mathbf{B}_\mathrm{e}]^{-1} + \mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}\}\delta\boldsymbol{x_a} = [\alpha\mathbf{B}_\mathrm{c} + (1-\alpha)\mathbf{B}_\mathrm{e}]^{-1}\delta\boldsymbol{x_b}+\mathbf{H}^T\mathbf{R}^{-1}[\boldsymbol{y} - \mathbf{H}\boldsymbol{x_t}]. \quad \text{A.12}$$

Multiplying each side of A.12 by its transpose, taking the expectation, assuming that observational errors are independent of background errors:

$$\{[\alpha\mathbf{B}_\mathrm{c} + (1-\alpha)\mathbf{B}_\mathrm{e}]^{-1} + \mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}\}\boldsymbol{A}\{[\alpha\mathbf{B}_\mathrm{c} + (1-\alpha)\mathbf{B}_\mathrm{e}]^{-1} + \mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}\}^T = [\alpha\mathbf{B}_\mathrm{c} + (1-\alpha)\mathbf{B}_\mathrm{e})]^{-1}E\{\delta\boldsymbol{x_b}(\delta\boldsymbol{x_b})^\mathrm{T}\}[\alpha\mathbf{B}_\mathrm{c} + (1-\alpha)\mathbf{B}_\mathrm{e})]^{-\mathrm{T}} + \mathbf{H}^T\mathbf{R}^{-1}\mathbf{R}\mathbf{R}^{-1}\mathbf{H}. \quad \text{A.13}$$

*Assuming the **B** contains the true background error covariances, i.e. the background errors are well specified, and using A.2:*

$E\{\delta\boldsymbol{x_b}(\delta\boldsymbol{x_b})^\mathrm{T}\} = \alpha\mathbf{B}_\mathrm{c} + (1-\alpha)\mathbf{B}_\mathrm{e}$

thus:

$$\{[\alpha\mathbf{B}_\mathrm{c} + (1-\alpha)\mathbf{B}_\mathrm{e})]^{-1} + \mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}\}\boldsymbol{A}\{[\alpha\mathbf{B}_\mathrm{c} + (1-\alpha)\mathbf{B}_\mathrm{e})]^{-1} + \mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}\}^T = [\alpha\mathbf{B}_\mathrm{c} + (1-\alpha)\mathbf{B}_\mathrm{e})]^{-1}[\alpha\mathbf{B}_\mathrm{c} + (1-\alpha)\mathbf{B}_\mathrm{e})][\alpha\mathbf{B}_\mathrm{c} + (1-\alpha)\mathbf{B}_\mathrm{e})]^{-\mathrm{T}} + \mathbf{H}^T\mathbf{R}^{-1}\mathbf{R}\mathbf{R}^{-1}\mathbf{H}. \quad \text{A.14}$$

or :

$$\{[\alpha\mathbf{B}_\mathrm{c} + (1-\alpha)\mathbf{B}_\mathrm{e})]^{-1} + \mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}\}\boldsymbol{A}\{[\alpha\mathbf{B}_\mathrm{c} + (1-\alpha)\mathbf{B}_\mathrm{e})]^{-1} + \mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}\}^T = [\alpha\mathbf{B}_\mathrm{c} + (1-\alpha)\mathbf{B}_\mathrm{e})]^{-1} + \mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}.$$

Dividing by $\{[\alpha\mathbf{B}_\mathrm{c} + (1-\alpha)\mathbf{B}_\mathrm{e})]^{-1} + \mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}\}$:

$$\boldsymbol{A} = \{[\alpha\mathbf{B}_\mathrm{c} + (1-\alpha)\mathbf{B}_\mathrm{e})]^{-1} + \mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}\}^{-1} \quad \text{A.15}$$

where $\boldsymbol{A} = E\{\delta\boldsymbol{x_a}(\delta\boldsymbol{x_a})^\mathrm{T}\}$ is the analysis error covariance matrix. A.15 demonstrates that independent forecasts updates in each ensemble member by using A.4 give the same optimal estimate of updated covariances as A.3.

Comment from Referee: 2. Equation (14).

It seems to me that it writes “innovation = innovation error”, which is wrong.

Author's response:

We thank the reviewer for this comment. We think our formulation is correct as the following is valid. Our eq.14 in the manuscript is:

$$d = [y - H(x_b)] = \varepsilon_o - (\varepsilon_r + \varepsilon_s)$$

where d *is the misfit,* ε_o *is the observational error,* ε_r *is the background random error and* ε_s *is the background systematic error.*

Introducing the true state of the ocean x_t*, (14) can be written also as:*

$$d = [y - H(x_b)] = y - H(x_t) + H(x_t) - H(x_b) = \varepsilon_o - H(\varepsilon_r + \varepsilon_s)$$

where the errors are defined as departures from the true state. If the observation network is dense, H ~ I, reducing to Eq. (14).

Author's changes in manuscript:

We will modify eq.14 in the manuscript including the intermediate equivalence to clarify.

Comment from Referee: 3. Ensemble update.

Due to the lack of rigorous formulation most hybrid methods employ empirical approaches for maintaining the ensemble spread. It seems that the manuscript does not tell explicitly how the ensemble members are updated. This is important for understanding the method and should be described. Further, on p. 10, l. 12-23 it is stated that the ensemble maintains spread due to observations and otherwise collapses due to the deterministic model forcing. This is somewhat contrary to what might be expected. It seems to me that increasing the number of observations in a consistent DA system should always reduce state error, that is always reduce the ensemble spread. Concerning the model forcing, in the context of a mainly stable forcing-driven model it is probably a pre-requisite to perturb forcing for ensemble members to match the corresponding uncertainty.

Author's response:

We fully agree that in a classical DA system increasing the number of observations should reduce the ensemble spread. This is due to the ensemble members’ generation procedure that usually involve perturbation in the initialization, surface and lateral open boundary conditions and in model physics (e.g. unresolved scales) as well.

In our experiment, differences between the ensemble members are generated perturbing only the observations, as explained in the manuscript p. 8 line 29 to page 9 line 13.

For the time being initialization, atmospheric forcing and lateral open boundary condition are unperturbed, the ensemble generation method spans the uncertainty linked with the observational sampling and assimilation formulation, implicitly acting on the background ensemble spread. This approach implicitly relies on a perfect model assumption, and it is likely to under-estimate the ensemble *covariances, implying that when no observations are assimilated, the spread equals 0 by construction, unlike most ensemble systems with full perturbations. By not perturbing the surface and lateral boundary conditions, we assume that the flow-dependent component of **B** is associated with the small-scale error fluctuations. Thus the deterministic large-scale forcing acts as an attractor for the ensemble perturbations, especially at the sea surface and in proximity of the boundaries. We plan in the future to relax this assumption and introduce of full set of perturbations spanning most of the uncertainties in the system.*

During the experiment we have assimilated a total of 3139 temperature and salinity vertical profiles deriving from gliders and CTD stations (255 CTD and 2884 Gliders vertical profiles). The perturbation of the observations, which is one the empirical aspects mentioned by the reviewer in his general comment of the manuscript, produces sensible differences in the observations and through differences in the solution of the minimization propagates in the model background of the following assimilation cycle. Each of the 14 ensemble members is an independent simulation with its own perturbation function and associated horizontal correlation radius, every assimilation cycle the observations differ but also the background of the individuals members are different, this produces the ensemble spread discussed and illustrated in Fig.4 and Fig.5. This method clearly strongly connects the growth of the ensemble spread to the perturbation function used and simultaneously link the ensemble spread to observations availability being otherwise forced by the same deterministic conditions (lateral and surface open boundaries).

Author's changes in manuscript:

We will include and mention more explicitly these issues in the revised version of the manuscript.

Comment from Referee: 3 Minor Issues

1. P. 2, l. 1: suggest replacing “not feasible to sample” by “not feasible to observe”.
2. P. 2, l. 24: suggest replacing “EnKF” by “traditional EnKF”.
3. P. 6, l. 19: suggest replacing “model bias error” by “model bias”.
4. P. 7, l. 3: suggest replacing “background error covariances” by “background errors”.

Author's response:

We thank the reviewer for these suggestions.

Author's changes in manuscript:

The revised version of the manuscript will be modified accordingly.

Comment from Referee: 4 Conclusion and recommendations

The manuscript addresses a difficult and interesting ocean forecasting problem. Some of the statements and approaches can be viewed as arguable (or, in regard to the EnKF, outdated), but this is what scientific literature is for. The methods used are in my view largely empirical, and again there is nothing wrong with that, as long this is clearly stated up-front. Concerning the results of the DA experiment described, they probably leave a lot of space for improvement, and this itself is one of the important outcomes of the manuscript. One line of statements that I tend to disagree with is that “it is difficult to run full EnKF with a large number of members” (p. 3, l. 14-17). Not in the year 2016, and definitely not with a 240 x 240 x 90 model. Overall, I believe that the manuscript will be interesting and useful for the ocean modelling and ocean forecasting communities. I recommend publishing it in Ocean Science after fixing the major issues listed above, which probably amounts to a major revision.

Author's response:

We thank the Reviewer for the general positive comments on the manuscript. In the revised version we tried to address all the major comments provided. Concerning the statements of p3. Line 14-17, this is written in the introduction and it is not explicitly linked to our experiment but rather to operational oceanography applications, however we will rephrase the sentence.

Author's changes in manuscript:

p3. Line 14-17 “Furthermore present computational resources limit the number of ensemble members accounted in operational EnKF.”

References

Wang, X., C. Snyder, and T. M. Hamill, 2007: On the theoretical equivalence of differently proposed ensemble-3DVAR hybrid analysis schemes. Mon. Wea. Rev., 135, 222–227.