Reviewer #1

While I am not overly enthusiastic about this paper, I believe that it would be suitable for publication after the grammar and sentence composition is improved.

Great efforts have been made to improve the grammar and sentence composition (see the makeup_revision (os-2016-15-supplement-version1.pdf)).

Reviewer #2

1. The title of the paper makes a clear reference to the Arctic Ocean, however the analysis of the instability presented here is not restricted to any specific part of the ocean. In particular, a direct quantitative comparison of the theoretical results to the intrusions in the Arctic Ocean is not extensive and occupies only a small fraction of the manuscript. I would thus recommend focusing one topic: either on application to the Arctic intrusions or on a discovery of a new type of fluid dynamical instability.

In view of this remark, we changed the title for “Generation of large-scale intrusions at baroclinic fronts: an analytical consideration with a reference to the Arctic Ocean”. The new title is focused on the analysis of the instability and, nevertheless, contains a reference to the Arctic Ocean. To my mind the reference allows to emphasize that (a) this study was primarily aimed to understand the nature of large-scale intrusions observed in the Arctic Ocean, and the aim was directly stated in the Introduction, (b) the approximations and simplifications used in the model, such as the weak geostrophic currents, the wide baroclinic fronts, the small Burger number, the small parameter beta, were fitted to the Arctic Ocean conditions (see Section 2), and (c) the theoretical results were compared with the observations of intrusions in the Arctic Ocean (see Section 3).

2. The author claims that this manuscript shows for the first time that a diffusion can destabilize the geostrophic flow. However, the novelty of the presented findings can be questioned as a discussion of the relevant scientific literature on viscous instabilities is not present. In particular, the author should discuss the work of McIntire (1970), Baker (1970), and Calman (1976) who consider experimentally and analytically the diffusive instability. In addition, a discussion of Murno (2010) experiments, which show the importance of viscosity, needs to be present. These are just several of the papers that came to mind, I’m sure there is more literature on the visco-diffusive instabilities of geostrophic flows.

You are absolutely right, drawing attention to a very important and delicate things. For clarity of presentation of the work more precise explanations are given now (p.8, lines 9-20; p.11, lines 19-27 in os-2016-15-manuscript-version3.pdf). First of all, I added to the manuscript an explanation of the differences between the McIntyre instability and instability obtained in this work (p.8, lines 9-20). For convenience, I reproduce some of the explanations here.

In accordance with (11) that was obtained from Eqs (1)-(10) at Bu<<1 (or at Ri>>1) and Pr=1, the large-scale disturbances can be unstable. Such instability has to be distinguished from the diffusive instability (McIntyre, 1970; Baker, 1970; Calman, 1976). Indeed, instability in the model by McIntyre occurs when Ri <(Pr + 1) ** 2 / 4Pr and is absent at Pr = 1.

The articles mentioned in your list either confirmed experimentally the McIntyre instability (Baker, 1970; Calman, 1976; Murno (2010)) or contained some useful additions to the McIntyre's theory (Calman, 1976). All of the mentioned articles were added to the References.

One of the important distinctions between these two models of baroclinic front instability is that in the present model the disturbances are allowed to have a nonzero slope in the along-front direction while in the model of diffusive instability by McIntyre (1970) the slope is taken zero.

Therefore, the McIntyre's model and other models in which the term $\partial p/\partial x\ \partial p/\partial x$ in the equations of motions is ignored (McIntyre, 1970; Calman, 1996; and others) can be referred as the 2D models (See also below the response to Comment No 9). Sometimes such models are called the models of symmetric instability.

From the mathematical point of view, the models that take into account the along-front slope of the perturbations, are much more complicated. Indeed, the analysis of the instability in the 2D models ultimately reduces to finding the roots of a polynomial depending upon the wave-number and growth rate. The models, that take into account the along-front slope of the perturbations, are reduced to the differential equations with variable coefficients, and such problems can be solved analytically only in rare cases.

3. Writing out the QG equations with the stratification parameter N that depends on y is not common. There should be a reference to a book or a paper that presents its proper derivation i.e. an asymptotic expansion in Ro number where N=N0+O(Ro). I'm not certain, but there might be some terms might be missing in Eq. 7 if N=N(y) – please check and give a reference.

It was said on p.3, lines 18-19 and 27 of os-2016-15-manuscript-version2.pdf that the dependence of the Brunt-Vaisala frequency upon the coordinate y is weak, that is, the inequality |s|fL << N0 ** 2 is satisfied. For this reason the additional terms in Eq. (7) can be neglected (it was pointed out on p. 3, line 27 of os-2016-15-manuscript-version2.pdf). A more detailed explanation of the issue is done in the revised manuscript (p.3, lines 25-26 of os-2016-15-manuscript-version3.pdf).

4. When neglecting the betta effect please provide quantitative estimates of a latitude at which betta-effect becomes less important (e.g. at 75 degree latitude beta is 25% of its value at the equator – is that beta negligible compared to shear term?). If betta effect from your scaling end up being negligible at any latitudes then the instability that you consider should be of a small horizontal length scale.

The value of parameter beta in the vicinity of intrusions observation is given on p. 10, lines 21-22 of the revised MS (os-2016-15-manuscript-version3.pdf). The comparison of the beta and shear terms is done on p.10, lines 27-28 of os-2016-15-manuscript-version3.pdf.

Of course, the larger the beta (i.e. the lower the latitude) the smaller horizontal length scale disturbances remain unaffected by the beta-effect. However, in the Arctic Ocean, near the Pole, beta is small, N/f~10, Bu<<1 for large-scale disturbances. The beta-effect term in Eq.(7) has approximately the same order of smallness as the relative vorticity, which is neglected in our model: Bu*U*k~beta*Bu/k. For this reason we can consider the large-scale (50-100 km) disturbances near the Pole.

In Section 3 of revised MS the values of all parameters are presented to confirm the correctness of our approach.

5. A discussion of why mass and momentum diffusivities are assumed to be the same is missing. Note, that in a non-turbulent regime, which might be adequate for the deep Arctic, the viscosity is an order of magnitude larger than heat diffusivity and three orders larger than salt diffusivity.

If the Eurasian Basin of the Arctic in the depth range of 600–1200 m is characterized by the non-turbulent regime, the existing models of interleaving will forecast the unstable modes with very small vertical length scale, which is obviously contrary to the observations. Merryfield (2002) suggested that this depth range is characterized by an intermittent turbulence and introduced a notion of differential mixing to parameterize the vertical diffusion terms. As a result, a satisfactory agreement between the vertical length scale of unstable modes and the thickness of observed intrusions in a purely thermohaline front has been achieved (a 3D model of the thermohaline front instability).

When the differential mixing parameterization was applied to the 2D model of the baroclinic front instability, a large difference was found between vertical scales of the unstable disturbances and the observed intrusions. A comprehensive discussion on the issue was presented in Kuzmina et al. (2014). If a suggestion on non-turbulent diffusivities is used in the 2D model of the baroclinic front instability, all unstable modes, including the maximum-growing one, will be of no more than several centimeters vertical scale.

This study suggests a weak turbulent rather than molecular regime in the deep Arctic layer under consideration, i.e. the difference between the momentum and mass exchange coefficients exists, but it is not high, obeying the condition of Pr*Bu<<1.

Much of what is said here, was presented in the previously submitted manuscript, but I still made some additions/explanations to the revised manuscript (p.10, lines 29-30 of os-2016-15-manuscript-version3.pdf).

*6. Deriving Eq. 9 from Eq. 8 requires vertical integration since p_z=-g rho and I'm not sure what was done with the vertical integral of the term U*p_zzx when U=U(z). Perhaps showing more steps would clarify things.*

The easiest way to show that Eq. 9 was derived from Eq. 8 correctly, is by differentiating Eq. 9 with respect to z for W0 = const (see the explanation on p. 4, lines 16-17 of os-2016-15-manuscript-version3.pdf).

Also, you can carry out the integration procedure of Eq. 8, applying the rule of integration by parts as follows:

$$\int \left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\left(\frac{\partial^2 p}{\partial z^2}\right)dz-\int vsfdz = K\frac{\partial^3}{\partial z^3}p+Const =$$

$$\frac{\partial}{\partial t}\partial p/\partial z+\int U\frac{\partial}{\partial x}\frac{\partial^2 p}{\partial z^2}dz-\int\frac{\partial p}{\partial x}sdz = K\frac{\partial^3}{\partial z^3}p+Const \quad . \qquad (8^*)$$

Here we took into account that $fv = \partial p/\partial x$.

Second term in the left of Eq. (8*) we can be rewritten using the rule of integration by parts as follows:

$$\int U\frac{\partial}{\partial x}\frac{\partial^2 p}{\partial z^2}dz = \frac{\partial}{\partial x}U\frac{\partial p}{\partial z}-\frac{\partial}{\partial x}psz+\int\frac{\partial p}{\partial x}sdz$$

Here we took into account that $U = sz^2/2$. Substituting the latter into Eq. (8*) we can obtain Eq. (9), remembering that $\partial p/\partial z = -g\rho$ and $v\partial\overline{\rho}/\partial y = vfsz/g$.

I doubt the need to include in the article as a detailed description of the integration procedure – I will do that if the Reviewer and the Editor consider it useful.

7. Eqns. 2 and 10 should contain the physical mechanism behind the instability which was not explained throughout the paper. The authors take a dry math approach to the instability problem by calculating the growth rates; however, omitting the physical mechanism of the instability dramatically reduces the understanding of the problem for the readers. Perhaps a schematic showing a positive feedback loop would be helpful. Also missing is a discussion of fundamental reasons for the existence of this instability. i.e. does it release a potential energy of mean flow, does it feed on its kinetic energy or something else?

In some cases, it is very difficult to elaborate the physical mechanism behind the instability. It is especially difficult in the case of the oscillatory instability. Remember the case of the instability of the critical layer: physicists could not explain it for a long time (only in 1975 Stern (1975) presented an interesting simple explanation of the critical layer instability).

With regard to our case of instability, it is possible to propose the following physical reasoning. As we can see from Eq. (16), the phase velocity of unstable disturbances is directed along the geostrophic current and exceeds the maximum velocity of the current. In such a case, in my opinion, the most likely is the conversion the kinetic energy of the main flow into the kinetic energy of disturbances.

The above-presented physical reasoning was included to the revised MS (p.8, lines 21-26 of os-2016-15-manuscript-version3.pdf).

8. In assessing the stability properties of Eq. 11 the author assumes a special case of U(z)_zˆ2 and present analytical solutions (Eq 15). It is not clear to me how was the growth rates in Eq. 16a,b calculated from Eq. 15.

It is explained in the revised MS (see p.7, line 2 of os-2016-15-manuscript-version3.pdf).

9. Eq. 11 that determines the stability of the flow does not have any y-derivatives and hence the stability properties do not depend on the y-direction wavelength. Thus, it looks like this instability is a 2D (in x,z–plane) rather than the 3D instability that the author claims to have investigated.

Eq. (7) is a 3D equation, and Eq. (11) which has no y derivatives was derived from Eq. (7) at Bu<<1. The absence of the y-derivative in Eq. (11) does not mean that the pressure disturbance does not depend on y because the velocity disturbance u is determined by Eq. (2). Moreover, we consider a finite width front with the length scale L along latitude (i.e. along the y co-ordinate). On the lateral boundaries of the front (y=0, L) the following conditions are met: v(y=0)=0, v(y=L)=0 (in accordance with Eq. (7)). For this reason, we have to seek the solution of Eq. (11) in the form (12): all decision variables depend of 3 co-ordinates x, y, and z. Thus, in my opinion, our model is a special (simplified) case of the 3D model describing the extra-long disturbances.

Some explanations were added to the revised MS in view of this remark (see p. 6, lines 3-5 of os-2016-15-manuscript-version3.pdf). Nevertheless, in order to avoid the 3D-2D confusion, in Abstract and Conclusions of the revised MS, the model is no longer directly referred as the 3D model.

10. The authors demonstrate that Eq. 11 has unstable solutions for U_zˆ2 but it is not obvious that other, more realistic, profiles of U(z) can also lead to an instability. It would be useful to solve Eq. 11 via the eigenvalue decomposition in z and show that arbitrary profiles of U (that have curvature in z) are indeed unstable.

To my mind, it would not be worth to consider here one more problem, which is much more complicated than the present one. The most interesting case is to consider the main flow velocity in the form of U = U1 + U2 + U3. Such problem is beyond the scope of this paper, and will be considered in the future. Here we draw attention to the importance of considering the parabolic shape of the main flow. Note that the effect of linear shear on the perturbation dynamics has not been analytically studied before.

11. The meaning of discussion of limits at z=+-inf is not clear and needs to be organized better. In particular at z=+-inf U->inf which is unrealistic for the ocean and for the theory which assumes U to be small in a QG sense. Why not using finite domain size z-=[0 H] and a corresponding no flux boundary conditions?

For the analysis of differential equations with variable coefficients it is necessary to use standard mathematical methods, which have been presented in this paper. However, to construct the solutions that are useful for applications it is needed to consider a layer of finite thickness - it was stated in the previous version of manuscript (see p.5, lines 22-31; p.7, lines 8-31of os-2016-15-manuscript-version2.pdf). In the revised MS, the additional explanations are presented (see p.7, lines 9-24, 29-30; p.8, lines 1-6 of os-2016-15-manuscript-version3.pdf).

12. It is not physical to have a growth rate that grows with increasing wavenumber as it implies that any kind of small scale noise would be preferentially amplified. The author motivates the paper with the idea that the size of intrusions in the arctic ocean might be explained by an instability. However, there seems to be no preferential wavelength at which the instability occurs and hence one cannot expect the appearance of intrusions of particular height. In addition, the theory breaks down at a particular length scale which the authors choose as the scale of intrusions and use it to calculate the growth rates. It is questionable to use these estimates since the theory technically does not apply at this marginal scales (i.e. the neglected terms need to be included).

The disadvantage of the model is its inability to forecast the characterictics of the most unstable mode – in the manuscript this issue has been discussed in detail (see p. 9, lines 11-17 of os-2016-15-manuscript-version2.pdf). However, it seems normal that the primary aim of the new instability problems is the proof of the potential for instability. For example, the pioneering work by Stern (1967), the first model of the DD interleaving, did not contain any estimate of the fastest growing mode because, like the present model, the growth rate increased unlimitedly with the wavenumber (it was discussed in os-2016-15-manuscript-version2.pdf too – see p.9, lines 11-17). On the other hand, in the studies of geophysical flow instabilities, some methods are used (e.g. the Rayleigh method) that can provide some conclusion about the possibility of instability, but cannot give the form of the unstable solutions and characteristics of the most unstable mode (remember e.g. the well-known work by J. Pedlosky(1964)).

The characteristics of the most unstable mode can be obtained by means of numerical integration of Eq. (7). Such a study being outside the scope of this paper is under way. However, the analytical considerations presented here showed that the unstable modes (not the fastest growing ones!) can occur at relatively large vertical wavelength of several tens of meters offering the principal possibility for explanation of the large-scale intrusions. Note that in all previous models of the baroclinic front interleaving, all unstable modes have had much smaller vertical wavelength.

13. Because there is no high wavenumber cutoff it is questionable whether the numerical model results shown in Fig. 1 are realistic; the step formation shown in Fig. 1a might be at the size of the numerical grid and hence their dynamics is not adequately resolved.

I'm sorry for the confusion - the small steps seen in Figs. 1 and 3 are an artefact caused by bad choice of the output data format used to store the results of numerical calculations. In the revised manuscript this annoying drawback has been corrected (see new Figs. 1 and 3).

14. A discussion of Orr-Sommerfeld equations seems unnecessary as it only makes a mathematical connection with insufficient improvement of our physical understanding of the problem; thus, it only makes the paper harder to understand.

I cannot withdraw fully the mentioning of the Orr-Sommerfeld equation because of the need to acquaint the reader with the important work by Miles (Miles, 1965).

Moreover, the analogies are often useful and can contribute to understanding the physics of the processes. Eq. (11) is a model (partial case) for the Orr-Sommerfeld equation. Also it is necessary to underline that the growing with time solutions are not relevant to the critical layer instability.

However, in accordance with this comment I cut the paragraphs devoted to the discussion of the O-S equation in the revised MS (see p.11, lines 23-35; p.12, lines 1-3 of the makeup revision (os-2016-15-supplement-version1.pdf)).

There are only a few sentences about the critical layer instability left in the revised MS (see p.8, lines 21-24 of os-2016-15-manuscript-version3.pdf).

15. Application to the Arctic Ocean can be questioned because i) there is no preferential length of instability that can be compared with the size of intrusions and ii) the growth rates are of the order of years are too large because the mean currents will most likely significantly change on the long time and very small spatial scales of the instability.

Strictly speaking, there are no models of baroclinic front instability that could fully describe the formation of large-scale intrusions in the Arctic Ocean at the stable-stable stratification (see Introduction where the issue is discussed in detail). In other words, the use of all existing theories can be questioned.

It is worth remembering that in this paper only a hypothesis on possible mechanisms of the large-scale intrusions generation in the Arctic Ocean is suggested.

It is important, that the new modes of instability have vertical scale that can reach tens of meters. However, the model is so complex that much more efforts are needed to obtain the exhaustive results of modelling which may be fully comparable with the empirical data. This paper is just the first step of the studies.

As to the growth time estimate of the order of years – I do not think it is too large. Contrary, it is in accordance with the results by Merryfield (2000, 2002) who resulted in the estimate of the time of formation of intrusions and the time scale of variability of the mean currents in the deep Arctic Ocean as several years.

16. I'd suggest working on the brushing up the grammar and logical presentation of the paper. Many paragraphs do not contribute well to the clarity of the paper and can be outright deleted. The title can be clearer as well: e.g. Generation of large-scale intrusions via diffusive instabilities.

Great efforts have been made to improve the grammar and logical presentation (see the makeup revision (os-2016-15-supplement-version1.pdf)).
Some paragraphs were outright deleted (see p.9, lines 10-21; p.10, lines 25-29; p.11, lines 13-16; p.11, lines 23-35; p.12, lines 1-3 of the makeup revision (os-2016-15-supplement-version1.pdf)).
The title has been changed.

Generation of large-scale intrusions at baroclinic fronts: an analytical consideration with a reference to the Arctic Ocean

~~A possibility of large scale intrusions generation in the Arctic Ocean under stable-stable stratification: an analytical consideration~~

Natalia Kuzmina

Shirshov Institute of Oceanology, 36 Nakhimovsky Ave, 117997 Moscow, Russia

Correspondence to: Natalia Kuzmina (kuzmina@ocean.ru)

Abstract. ~~Some a~~Analytical solutions are found for the problem of ~~three-dimensional~~ instability of a weak geostrophic flow with linear velocity shear ~~taking into~~ accounting for vertical diffusion of buoyancy. The analysis is based on the potential vorticity equation in a long-wave approximation when the horizontal scale of disturbances is ~~taken to be~~considered much larger than the local baroclinic Rossby radius. It is hypothesized that the solutions found can be applied to describe stable and unstable disturbances of the ~~on a~~ planetary scale with respect, especially, to the Arctic Ocean~~Basin~~, where weak baroclinic fronts with typical temporal variability periods of the order of several years or more ~~are~~ have been observed and the β ~~beta~~-effect is negligible. Stable (decaying with time) solutions describe disturbances that, in contrast to the Rossby waves, can propagate to both the west and east, depending on the sign of the linear shear of geostrophic velocity. The unstable (growing with time) solutions are applied to explain the formation of ~~describe~~ large-scale intrusions at baroclinic fronts under the stable–stable thermohaline stratification observed in the upper layer of the Polar Deep Water in the Eurasian Basin. The ~~proposed~~ suggested mechanism of formation of intrusions ~~description of intrusive layering~~ can be considered ~~as~~ a possible alternative to the mechanism of interleaving at the baroclinic fronts due to the differential mixing.

1 Introduction

The ~~S~~study of intrusions in oceanic frontal zones is required to understand ~~necessary to analyse~~ the mechanism~~s~~ of ventilation and mixing in the ocean interior (see, for example, Zhurbas et al., 1983, 1987; Rudels et al., 1999, 2009; Kuzmina and Zhurbas, 2000; Walsh and Ruddick, 2000; Merryfield, 2000; Radko, 2003; Richards and Edwards, 2003; Kuzmina et al., 2005, 2011; Smyth and Ruddick, 2010). Intrusive layering, as a rule, results from the instability of oceanic fronts. One of the major mechanisms responsible for the instability of both thermohaline and baroclinic fronts is related to the double diffusion (Stern, 1967; Ruddick and Turner, 1979; Toole and Georgi, 1981; McDougall, 1985a, 1985b; Niino, 1986; Yoshida et al., 1989; Richards, 1991; Kuzmina and Rodionov, 1992; May and Kelley, 1997; Kuzmina, 2000). However, in the Eurasian Basin of the Arctic Ocean there are baroclinic and thermohaline fronts within the upper layer of the Polar Deep Water (PDW) populated with intrusive layers of vertical length scale as large as 30 m and with horizontal scale reaching up to more than 100 km (Rudels et al., 1999, 2009; Kuzmina et al., 2011) observed at the stable–stable stratification (i.e., ~~when~~ for the mean salinity increasing ~~es with depth~~ while the mean temperature decreasing~~es~~ with depth). It can be suggested that the thermohaline intrusions within the upper layer of PDW are driven by differential mixing. Merryfield (2002) was the first to show satisfactory agreement between calculations of unstable modes from a 3D interleaving model that ~~, taking into~~ accounted for the differential mixing at a non-baroclinic~~ity~~ front and observations of intrusive layering at a pure thermohaline front in the PDW. Merryfield's (2002) findings were confirmed by Kuzmina et al. (2014). However, the 2D model of interleaving driven by differential mixing at the baroclinic front failed to simultaneously fit ~~did not show a satisfactory fit simultaneously between the~~ three modelled parameters, namely the vertical scale, the growth time and the slope of the fastest growing mode with ~~, and~~ observations of intrusions in a frontal zone with a substantial baroclinicity in the upper PDW layer (Kuzmina et al., 2014). In particular, it was found that the vertical scale of the most unstable mode was about ~~is~~ two to three times smaller than the vertical scale of intrusions observed in the baroclinic front. Furthermore, it is worth noting that the 2D models of double-diffusive interleaving, as applied to typical baroclinic fronts in the ocean, are able to forecast intrusive layers with vertical length scale of no more than 10 m ~~vertical length scale~~ (Kuzmina and Rodionov, 1992; May and Kelley, 1997, 2001; Kuzmina and Zhurbas, 2000; Kuzmina and Lee, 2005; Kuzmina et al., 2005). Therefore, despite the ~~fact that there are~~ proven-by-simulation hypothesis of intrusions ~~es of a merger~~ of small vertical scale ~~intrusions~~ merging into larger structures (Radko, 2007), new approaches to the mathematical description of the formation of large intrusions in the area~~s~~ of baroclinic fronts become ~~appear~~ relevant.

We suggest that the interleaving at a baroclinic front~~s~~ may be considered as a result of 3D instability of weak geostrophic current due to the combined effects of vertical shear and diffusion of density (buoyancy).

The effect of vertical diffusion of buoyancy on the baroclinic instability of geostrophic zonal wind has been ~~was~~ studied theoretically by Miles (Miles, 1965). Proceeding from the ~~Based on an~~ analogy between the equations describing the dynamics of large-scale atmospheric perturbations and the Orr–Sommerfeld equation (Lin, 1955; Stern, 1965), Miles (1965) analyz~~s~~ed the instability of the critical layer (a ~~the~~ very thin layer in which the phase velocity of a disturbance ~~is~~ equals ~~to~~ the velocity of zonal flow). He resulted in ~~As a result, Miles built~~ an analytical asymptotic solution ~~taking into~~ accounting for a ~~the~~ very small, though ~~but~~ finite, vertical diffusion of buoyancy. Based on the analysis, Miles (1965) concluded that the effect ~~influence~~ of vertical diffusion of buoyancy on the destabilisation of ~~in destabilising the~~ zonal wind is negligible ~~not essential~~ in comparison with the baroclinic instability (the generation of cyclones and anticyclones) for typical atmospheric geostrophic winds. One could ~~can~~ assume, however, that other conditions ~~situations~~ can occur ~~be observed~~ in the deep ocean. Indeed, in the Polar zones, for example, in the Eurasian Basin of the Arctic, very weak geostrophic currents have been ~~are~~ observed in ~~at~~ deep layers (Aagaard, 1981). These currents can have a large horizontal (transverse) scale and large time scale of variability, the latter being estimated to exceed ~~at much more than~~ one year (Aagaard, 1981). Taking into account, that the influence of the β-effect on the dynamics of large-scale disturbances is negligible in the Polar Ocean, it seems reasonable to suggest, that the contribution ~~role~~ of diffusion of buoyancy to ~~in~~ the destabilization of weak geostrophic currents can be important. Therefore, in such circumstances one would expect the formation of intrusions, rather than vortices.

The present work is devoted to seeking ~~the search for~~ analytical unstable (increasing with time) and stable (decreasing with time) solutions based on the potential vorticity equation describing the 3D dynamics of a weak baroclinic front, with the vertical diffusion of buoyancy included. Hopefully t~~T~~he results~~, hopefully,~~ will provide an opportunity ~~make it possible~~ to obtain some new insight into ~~conceptions about~~ the causes of the formation of large intrusions, particularly in the regions of the Arctic Ocean ~~Basin~~ with the stable–stable stratification.

2 Problem formulation, derivation of basic equation, and solution search

Let us consider the problem of the 3D instability of a ~~the~~ baroclinic front based on ~~the basis of~~ the linearized equations of motion in quasi-geostrophic approximation (see, for example, Pedlosky, 1979; Cushman-Roisin, 1994):

$$U = -\frac{1}{f}\frac{\partial P}{\partial y},\ V = 0,\ \ W = 0,\ \frac{\partial P}{\partial z} = -g\overline{\rho} \tag{1}$$

$$\frac{\partial p}{\partial z} = -g\rho,\ u = -\frac{1}{f}\frac{\partial p}{\partial y},\ v = \frac{1}{f}\frac{\partial p}{\partial x} \tag{2}$$

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\left(\frac{1}{f}\Delta p\right) + \beta v - f\frac{\partial w}{\partial z} = \frac{\tilde{K}}{f}\frac{\partial^2}{\partial z^2}\Delta p \tag{3}$$

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\rho + v\frac{\partial\overline{\rho}}{\partial y} - \frac{N^2}{g}w = K\ \frac{\partial^2}{\partial z^2}\rho, \tag{4}$$

where U and V are zonal and meridional components of the geostrophic velocity, P and $\overline{\rho}$ are the mean pressure and density both divided by the reference density, *N*, *f* and *g* are the buoyancy frequency, Coriolis parameter and gravity acceleration, u, v and w are velocity fluctuations along the *x*, *y* and *z* axes, respectively, p and ρ are the pressure and density fluctuations both divided by the reference density, $\beta = \partial f / \partial y$, $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$, and the *x*, *y* and *z* axes are directed eastward, northward and upward, respectively. The ~~V~~vertical friction with a constant coefficient $\tilde{K}$ is considered in the vorticity equation Eq. (3). The density balance equation Eq. (4) takes into account, apart from the advection terms, ~~advection Eq. (4) accounts for~~ only the vertical diffusion with a constant coefficient K. The constant coefficients $\tilde{K}$ and K are treated as the average values over the ~~in an~~ ocean layer under investigation.

Let us take the distribution of mean density, divided by the reference density, ~~normalized to the reference density,~~ as follows:

$$\overline{\rho}(z, y) = fsyz/g + f\tilde{s}y/g - \frac{N_0{}^2}{g}z + 1, \tag{5}$$

where $N_0 = const > 0$ is a characteristic value of the buoyancy frequency in the frontal zone, and $\tilde{s}$ and s are dimensional constants, either positive or negative, that characterize the cross-front gradient~~s~~ of density and the vertical shear of the basic geostrophic flow.

The first term on the right of Eq. (5) has not been taken into account in the interleaving models describing both the 2D (see, for example, Kuzmina, Rodionov, 1992; May, Kelley, 1997; Kuzmina, Zhurbas, 2000) and 3D (Eady, 1949; Miles, 1965; Smyth, 2008) instabilities of the oceanic baroclinic fronts. Meanwhile, ~~However,~~ the oceanic fronts can be characterized by not only the cross-front gradient of density, but also by the cross-front gradient of the buoyancy frequency. This is the case described by Eq. (5): the squared buoyancy frequency~~,~~ $N^2 = -gd\overline{\rho}/dz$, is a linear function of *y*. This dependence is assumed to be weak: $|s|fL << N_0{}^2$, where *L* is the characteristic lateral length scale (width) of the frontal zone ($0 \le y \le L$). However, even a weak lateral change in the buoyancy frequency indicates the existence of a quadratic dependence of geostrophic velocity on the vertical coordinate *z*. Indeed, if the mean density distribution is expressed by Eq. (5), the geostrophic current velocity will be

$$U = U_1 + U_2 + U_3,\ U_1 = sz^2/2,\ U_2 = \tilde{s}z,\ U_3 = const, \tag{6}$$

where U_1 and~~,~~ U_2 are the constituents of geostrophic velocity with linear (U_1) and constant (U_2) vertical shear: $dU_1/dz = sz$, $dU_2/dz = \tilde{s}$ and~~;~~ U_3 is the barotropic (constant) velocity addition.

The equation of evolution of potential vorticity, derived from ~~on the basis of~~ Eqs. (1~~)~~–~~(~~4) under the assumptions of $\left|fsy/N_0{}^2\right| \le \left|fsL/N_0{}^2\right| << 1$ and $U/fL << 1$, is

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\left(\frac{\partial^2 p}{\partial z^2} + \frac{N_0{}^2 \Delta p}{f^2}\right) + \frac{\beta v N_0{}^2}{f} - vsf = K\frac{\partial^4}{\partial z^4}p + \tilde{K}\frac{N_0{}^2}{f^2}\frac{\partial^2}{\partial z^2}\Delta p \qquad . \tag{7}$$

~~Equation (7) was derived under the abovementioned assumption of~~ $\cancel{\left|fsL/N_0{}^2\right| << 1}$~~.~~ Note that ~~in~~ the differentiation of Eq. (4) with respect to z~~,~~ cancels out the terms ~~such members as~~ $(\partial U/\partial z)\cdot(\partial\rho/\partial x)$ and $(\partial v/\partial z)\cdot(\partial\bar{\rho}/\partial y)$ ~~are reduced~~, since according to Eqs. (1) and (2) they are equal in magnitude and opposite in sign~~, in accordance with Eqs. (1) and (2)~~.

As it can be seen from Eq. (7), the last term on the left can strengthen or weaken, depending on the sign of s, the impact of the β-effect on the dynamics of disturbances.

We will consider at $K \approx \tilde{K}$ the long-wave disturbances (i.e. perturbations of the ~~on a~~ planetary scale) of weak geostrophic current ~~(~~ $\cancel{F(z) = B_1F_1(z) + B_2F_2(z) + B_3F_3(z)}$ ~~)~~ which satisfy the following relationship between the vertical and horizontal length scales (H and $\tilde{L}$, respectively): $\tilde{L} >> L_R$, where $L_R = N_0 H/f$ is the baroclinic Rossby radius of deformation. If we apply Eqs. (1–4) to describe the motion in the Arctic Basin, the β-effect term can be ignored because $\beta \approx 0$ in the vicinity of the North Pole.

Taking into account the abovementioned conditions, we may use the method of series expansion at small parameter $\delta^2 = N_0{}^2 H^2/(\tilde{L}^2 f^2) = Bu$, where Bu is the Burger number (see e.g.~~, for example,~~ Cushman-Roisin, 1994). For~~At~~ $\delta^2 \sim 10^{-3} - 10^{-4}$ ($\tilde{L} \sim 10^4 - 10^5$ m, $20 < H < 100$ m, $N_0 \sim 10^{-3}\,\mathrm{s}^{-1}$, $f \sim 10^{-4}\,\mathrm{s}^{-1}$), it is reasonable to consider only the first term of the series. In this case, we can rewrite the potential vorticity equation in the simplified form:

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\left(\frac{\partial^2 p}{\partial z^2}\right) - vsf = K\frac{\partial^4}{\partial z^4}p\,. \tag{8}$$

The introduced by the procedure relative error ~~committed in~~ of the solution ~~by so doing~~ is expected to be of the order of δ^2, and the smaller δ^2 ~~,~~ the smaller the error.

According to ~~In accordance with~~ our approximation, Eq. (8) corresponds to the density balance ~~advection~~ equation Eq. (4) for~~at~~ $w = w_0 = const$:

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\rho + v\frac{\partial\bar{\rho}}{\partial y} - \frac{N_0{}^2}{g}w_0 = K\frac{\partial^2}{\partial z^2}\rho\,. \tag{9}$$

The correspondence between Eqs. (8) and (9) can be checked by differentiating Eq. (9) with respect to z and taking into account that $\partial p/\partial z = -g\rho$.

Thus, the vorticity equation Eq. (3) drops out of consideration. Indeed, given that the diffusivity of mass, K, in the oceanic interior (particularly in the deep water of the Arctic Ocean) probably does not exceed the value of 1×10^{-5} m^{2}s^{-1} and the vertical length scale of the intrusions, H, ~~to~~ which this theory is applied to, is approximately equal to $H \approx 20-100$ m, the ratio of $U/\tilde{L}$ is estimated as $\cancel{U/\tilde{L} \le 10^{-8}}$ $U/\tilde{L} < 3 \cdot 10^{-8}$ s^{-1}. Based on the latter estimate, one can suggest that the vertical circulation caused by the frictional force and temporal change of vorticity, ~~and frictional force~~ will not significantly affect the dynamics of large-scale disturbances. This hypothesis will be tested a posteriori by analysing the solutions obtained. For further discussion it is important to underline that the geostrophic Richardson number $Ri = N_0{}^2 H^2 / \tilde{U}^2$, where $\tilde{U}$ is the characteristic scale of geostrophic velocity, is much larger than unity for very slow currents.

Given that $w_0 = const$, we can take ~~put~~ $w_0 = 0$ in Eq. (9), and therefore rewrite it ~~Eq. (9)~~ as

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\rho + v\frac{\partial \overline{\rho}}{\partial y} = K\frac{\partial^2}{\partial z^2}\rho \,. \tag{10}$$

Based on the reasoning above, we can conclude that the slow extra-large-scale disturbances of weak geostrophic flow are described by the quasi-stationary system of Eqs. (2) and (10).

Let us now pay attention to an important issue. Namely, if we suppose that $U(z) = 0$ in Eq. (10) and consider salt fingering instead of diffusion of buoyancy, then, in addition to Eq. (2), it will be necessary to write the following two equations instead of Eq. (10):

$$\frac{\partial \rho}{\partial t} = K_S (1-\gamma)\tilde{\beta}\frac{\partial^2 S}{\partial z^2}\,, \quad \frac{\partial S}{\partial t} + v\frac{\partial \overline{S}}{\partial y} = K_S\frac{\partial^2 S}{\partial z^2}\,, \tag{10´}$$

where S and $\overline{S}$ are the salinity disturbance and mean, K_S and $\gamma = \tilde{\alpha}F_T / \tilde{\beta}F_S < 1$ are the vertical diffusivity of salinity and the flux ratio for salt finger convection, F_T and F_T are the vertical fluxes of temperature and salinity, $\tilde{\alpha}$ and $\tilde{\beta}$ are the temperature expansion and salinity contraction coefficients, respectively.

Eq~~uations~~s. (10´) along with (2) constitute the system of equations that was used by Stern (1967) to obtain the polynomial dependence between the growth rate of unstable perturbations, wave numbers and hydrological parameters [see Eq. (4) of (Stern, 1967)]. Therefore, the proposed model, which consists of Eqs. (2) and (10), can in a certain sense be regarded as an analogue of the model by Stern (1967) for investigating the interleaving on a large horizontal scale. ~~However, the derivation of the model equations, which we have done above, is useful to understand the limits of the model's applicability.~~

From the point of view of the author of this ~~work~~paper, a simple quasi-stationary (geostrophic) system of equations accurately describes the large scale movement especially in the Arctic Ocean, where the influence of the ~~beta~~ β-effect is not significant, the baroclinic fronts of large width in the ocean interior are often not intense (Kuzmina et al., 2011), and the baroclinic radius of deformation, HN/f, at $H \sim 100$ m, does not exceed 2–5 km (see also Section 3).

To analyse the instability of the geostrophic flow in the frame of Eqs. (2) and (10), let us take ~~consider~~ a layer with the~~a~~ vertical scale of $2H_0$, move the *z*-axis origin to the middle of the layer, and ~~place the co-ordinate system on the middle line of the layer. For the analysis of the instability in the frame of Eqs. (2) and (10), we will~~ consider a symmetric relative to the midline ~~of the layer~~ geostrophic flow with quadratic *z*-dependence of velocity:

$$\cancel{U = U_1 + U_3 = sz^2/2 + U_3}\ U = U_1 + U_3\,,\ U_1 = sz^2/2\,,\ U_3 = -\mathrm{sign}(s)\cdot sH_0{}^2/2\,.$$

A parabolic z-dependence of the geostrophic flow velocity ~~upon the vertical co-ordinate~~ can be observed in the rotary flow of the intra-pycnocline vortices, as well as in many other ocean flows. In any case, as mentioned above, in the oceanic frontal zones it is ~~not un~~likely to observe changes of the buoyancy frequency in the cross-front direction indicating the presence~~, which indicate the existence~~ of linear shear of geostrophic velocity. The ~~C~~consideration of the instability of geostrophic flow~~ instability~~ with the velocity profile of $U = U_1 + U_2 + U_3$ is also possible~~ on the basis of~~ by analytical methods, but this issue falls out of the scope of the present study.~~ and we will consider the related issues below.~~

Let us discuss the conditions on the boundaries of the layer in relation to the ocean. Keeping in mind the Eady problem (Eady, 1949), one has to set ~~require~~ the ~~vanishing of~~ vertical velocity vanishing at the layer boundaries. ~~In accordance with o~~Our approximation meets~~,~~ this condition~~ is satisfied~~.

Due to ~~the fact that the~~our model ~~takes into account~~accounting for the vertical diffusion, it appears reasonable to accept ~~is logical to take~~ the conditions of zero buoyancy flux (for density perturbations) at the layer boundaries: $p_{zz} = 0$ at $z = \pm H_0$ (the type 1 boundary conditions). It is reasonable to consider another type of condition too, namely, the slippery boundary conditions or equivalent~~ conditions of the vanishing~~ of density disturbances vanishing at~~ the~~ boundaries: $dv/dz = du/dz = \rho = 0$ at $z = \pm H_0$ (the type 2 boundary conditions). Under the type 2 boundary conditions, it is necessary to set up ~~require~~ the absence of convergence or divergence of buoyancy flux within the layer: $p_{zz}(z = H_0) = p_{zz}(z = -H_0)$. This condition ~~requirement~~ is necessary, because the convergence or divergence of the buoyancy flux within the layer may increase or, conversely, decrease the stability of the layer.

Using Eq. (2), we rewrite Eq. (10) as:

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\left(\frac{\partial p}{\partial z}\right) - \frac{\partial p}{\partial x}sz = K\frac{\partial^3}{\partial z^3}p\,, \qquad (11)$$

where $U = U_1 + U_3$.

To analyse the instability of geostrophic flow, we will seek the solution of Eqs. (2) and (11) ~~Eq. (11) at $L = \tilde{L}$~~ for the lateral boundary conditions of $v|_{y=0} = v|_{y=L} = 0$, accordingly to Eq. (7), in the form

$$p = \mathrm{Re}\left\{F(z)e^{ik(x-ct)}\sin(\pi y/L)\right\}, \qquad (12)$$

where k is the wave number along the x axis and c is the growth rate. The ~~D~~disturbances of the horizontal velocities will be expressed as $u = -p\pi/Lf$ and $v = pik/f$. ~~The solution will be unstable, i.e., increasing with time, if the~~ For the positive imaginary part of c~~ is positive:~~, $\mathrm{Im}(c) > 0$, the solution will be unstable, i.e. increasing with time.

The substitution of Eq.~~Substituting~~ (12) into Eq. (11) yields the following equation:

$$ik\left(U_1 + U_3 - c\right)\left(\frac{dF(z)}{dz}\right) - F(z)iksz - K\frac{d^3}{dz^3}F(z) = 0\,. \qquad (13)$$

We are interested in finding an answer to the following question: is it possible to make certain judgements about the possibility of instability of geostrophic flow in a finite vertical layer, based on the analytical solutions of Eq. (13) at some values of parameter c?

It is easy to verify that the following functions are ~~the~~ partial solutions of Eq. (13):

$$F_1(z)=e^{-az^2/2}\,,\,F_2(z)=az^2+D\,, \tag{14}$$

where $a^2=iks/2K$, $D=-2a(c-U_3)/s$, $ikc=ik(c_1+ic_2)=5a\cdot K+ikU_3$, $c_1=\mathrm{Re}\,c$, $c_2=\mathrm{Im}\,c$, and $U_1+U_3-\mathrm{Re}\,c\neq 0$ at~~for~~ an arbitrary point $z=z_0$ ~~lying inside~~ in the layer domain.

To test partial solutions Eq. (14), one has to substitute $F_1(z)$ and $F_2(z)$ from Eq. (14) into Eq. (13), reduce the latter to a cubic polynomial $P(z)=A_3z^3+A_2z^2+A_1z^1+A_0z^0$ and evaluate ~~the~~ coefficients A_0, A_1, A_2, and A_3. It is easy to make sure, ~~obtain~~ that this polynomial is identically zero (i.e.~~,~~ $A_0\equiv 0$, $A_1\equiv 0$, $A_2\equiv 0$, and $A_3\equiv 0$).

~~Based on~~Proceeding from the theory of ordinary differential equations (see, for example, Polyanin and Zaitsev, 2001), due to the linearly independent ~~and taking into account that the~~ functions $F_1(z)$ and $F_2(z)$ ~~are linearly independent~~, we can express ~~write~~ the general solution of Eq. (13) for $ikc=5a\cdot K+ikU_3$ in the form:

$$\begin{aligned}&F(z)=B_1F_1(z)+B_2F_2(z)+B_3F_3(z)\\&F_3(z)=F_1(z)\cdot\int F_2(z)\varphi(z)dz-F_2(z)\cdot\int F_1(z)\varphi(z)dz\,,\\&\varphi(z)=(F_1(z)\cdot dF_2(z)/dz-F_2(z)\cdot dF_1(z)/dz)^{-2}\end{aligned} \tag{15}$$

where B_1, B_2, B_3 are arbitrary constants. It is important to note two facts. Firstly, the functions $F_1(z)$ and $F_2(z)$ are even functions, while $F_3(z)$ is an odd function. Secondly, despite the singularity ~~at $z=0$ in the~~ of integrands ~~of~~ in Eq. (15) at $z=0$, the function $F_3(z)$ is differentiable at this point. (The latter becomes evident from the asymptotic analysis ~~last can be seen by analysing the behaviour~~ of function $F_3(z)$ ~~when~~ for $z\to 0$.)

Let us now consider the unstable and stable solutions Eq. (15).

2.1 Unstable solutions

According to the expression for parameter *c* (see Eq. (14)), $ikc=ik(c_1+ic_2)=5a\cdot K+ikU_3$, ~~S~~solution Eq. (15) can be unstable for both $s>0$ and $s<0$. The real and imaginary parts of kc for the unstable (i.e. growing with time) solutions ~~growing with time (unstable)~~ are

$$kc_1=2.5\cdot\sqrt{|s|kK}+kU_3,\; kc_2=2.5\cdot\sqrt{|s|kK}\quad \text{~~at~~for } s<0\,, \tag{16 a}$$

$$kc_1=-2.5\sqrt{skK}+kU_3,\; kc_2=2.5\cdot\sqrt{skK}\quad \text{~~at~~for } s>0\,. \tag{16 b}$$

~~Formulas~~ Eqs. (16) demonstrate that the condition $U_1+U_3-\mathrm{Re}\,c\neq 0$ is satisfied for $z\in(-\infty,+\infty)$.

According to Eqs. (16), ~~the growth rate increases with the increase in *s* and *K*, which implies that not only double diffusion, but the diffusion of buoyancy can cause instability of the geostrophic flow. However,~~ the unstable solution is realized ~~at~~ for $\mathrm{Re}\,a<0$, and hence, for any finite wave number k, the function $F_1(z)$ and all its derivatives increase infinitely, if ~~dramatically when~~ $z\to\pm\infty$. On the other hand, the function $F_3(z)$ and all its derivatives ~~decrease when~~ vanish, if $z\to\pm\infty$. (~~This~~ It can be seen ~~by analysing the~~ from asymptotic analysis ~~behaviour~~ of the integrals that define ~~defining function~~ $F_3(z)$, if ~~when~~ $z\to\pm\infty$.) Therefore, to prove the instability in a finite layer, it is necessary to show that $F(z)$ ~~at~~ for $\mathrm{Re}\,a<0$ is an eigenfunction of the ~~proper~~ eigenvalue problem with the boundary conditions of type 1 or 2 introduced above. To construct physically correct solutions we will consider the following two cases. Case 1, ~~when~~ where the vertical scale of the layer corresponds to our approximation: $2H_0 \sim H << fL/N$. Case 2, ~~when~~ where the vertical scale of the layer significantly exceeds the vertical scales of the disturbances for which our approximation holds true: $H << 2H_0$.

To satisfy the boundary conditions of type 1 and 2 ~~(either type 1 or type 2)~~ in case 1, we have to take $B_3 = 0$, because $F_3(z)$ is an odd function. The type 1 boundary conditions are reduced to ~~the following conditions for~~ ~~$F(z)$:~~ $F_{zz} = 0$ ~~at~~ for $z = \pm H_0$. Thus, the following equality should be ~~satisfied~~ met:

$$e^{-aH_0^2/2}(-1 + aH_0^{\ 2}) + 2B_2/B_1 = 0\,. \tag{17}$$

Given that $2B_2/B_1$ can have different values, the instability in the framework of solution Eq. (15) does exist, because in a wide range of typical ocean values of H_0, s, and K, there is a ~~the~~ wave number $k_0 << f(2N_0H_0)$ ~~at~~ for which Eq. (17) is satisfied.

The type 2 boundary conditions are reduced to $F_z = 0$ at $z = \pm H_0$. Under such conditions, the requirement of the absence of the buoyancy flux convergence/divergence within the layer is ~~satisfied~~met: in ~~the~~ case of parity of $F(z)$ for $B_3 = 0$ and for the flow symmetry ~~a flow that is symmetric~~ relative to the midline ~~of the layer and the parity of function (15)~~, the values of buoyancy flux at the boundaries are of the same magnitude and direction (sign). Under the type 2 boundary conditions the following equality should be ~~satisfied~~met:

$$e^{-aH_0^2/2} - 2B_2/B_1 = 0\,. \tag{18}$$

Obviously, in this case, as in the case of Eq. (17), there is a wave number $k_0 << f/(2N_0H_0)$ ~~at~~ for which Eq. (18) is satisfied.

Figure 1 presents ~~For case 1,~~ graphic images of ~~the~~ unstable solutions in the form of density disturbances ~~corresponding to the disturbances of density~~ $\mathrm{Re}\,\rho = \mathrm{Re}(dF/dz) = \tilde{\rho}$ for different boundary conditions for the case 1. ~~are presented in Fig. 1.~~ When building the solutions, typical values of hydrological parameters in relation to the Arctic Basin were used (see Subsection 2.3 and Section 3).

In the case 2, we have to take $B_1 = 0$, $B_2 = 0$, and consider $F_3(z)$ as the solution of the eigenvalue problem. Indeed, $F_3(z)$ and all its derivatives sharply decrease, if ~~when~~ $z \to \pm\infty$, and, consequently, on the boundaries of the large vertical scale layer the function $F_3(z)$ and all its derivatives should be infinitesimally small. Therefore, ~~that is,~~ the boundary conditions of type 1 and 2 are satisfied. Indeed, the characteristic vertical scale of the decrease of $F_3(z)$ at $z \to \pm\infty$ can be evaluated as $h \sim (ks/K)^{-1/4}$. If H_0 is n times as large as h, the value of $|F_3(z)|$ at $|z| = H_0$ will be e^{n^2} times (!) as small as the maximum value of $|F_3(z)|$. Note, that the maximum value of geostrophic velocity $U_{\max}$ increases with H_0. However, $U_{\max}$ should satisfy the condition of $U_{\max}k/f << 1$, which can be rewritten as $n^2(ksK)^{1/2}/f^2 << 1$. It is easy to see that this condition is met in a wide range of k, s and K for $n = 5$–10.

A plot of the disturbances of density $\tilde{\rho} = \mathrm{Re}(dF_3/dz)$ corresponding to unstable solutions is presented in Fig. 2. It is worth noting that the function $\rho = dF_3/dz$ is differentiable at $z = 0$ likewise the function $F_3(z)$. ~~In this case, it is relatively simply to construct an analytical solution for a more complicated form of geostrophic current such as $U = U_1 + U_2 + U_3$. To make it, we have to seek solutions of (13) (preliminarily rewriting this equation for $U = U_1 + U_2 + U_3$) in the form:~~

$$\cancel{\tilde{F}_1(z) = \exp(-az_*^2/2),\ \tilde{F}_2(z) = az_*^2 + D,\ z_* = z + \tilde{s}/s,\ D = -2a(c - U_3 + \tilde{s}^2/2s)/s.}$$

~~The function $\tilde{F}_3(z)$ is constructed similarly to previous considerations (see formula (15)). This function will have an additional oscillating component in comparison with function $F_3(z)$.~~

~~For case 2, a plot of the unstable solutions corresponding to the disturbances of density $\tilde{\rho} = \mathrm{Re}(dF_3/dz)$ is presented in Fig. 2. It is worth noting that the function $\rho = dF_3/dz$ is differentiable at $z = 0$ as well as the function $F_3(z)$.~~

~~Thus, if the vertical diffusion of buoyancy plays a role in the dynamics of ocean processes, the long-wave perturbations of the weak baroclinic front with linear shear can be unstable (time-increasing). Note that in the case of no baroclinicity, instability due to the diffusion of buoyancy cannot arise. Indeed, when $U = 0$, Eq. (13) becomes the diffusion equation, and its general solution for the density perturbation (12) at $c \neq 0$ has the form (see, for example, Niino, 1986):~~

$$\cancel{F'_z = \varphi(z) = Pe^{nz} + Qe^{mz},}$$

~~where P and Q are integration constants, while n and m are expressed as $n^2 = m^2 = (-ikc/K)$, $n = +(-ikc/K)^{0.5}$, $m = -(-ikc/K)^{0.5}$.~~

~~With regard to the boundary conditions, the problem is reduced to an eigenvalue problem. Here, we briefly discuss only the evaluation of the growth rate c_2. It can be easily shown that the solution $\varphi(z)$ will satisfy the abovementioned boundary conditions only if $e^{4nH_0} = 1$. This equality can be satisfied if n is zero or an imaginary number. The imaginary value of n indicates that the value of the growth rate c_2 is negative for all values of the wave number k. In this case, the disturbance is described by trigonometric functions that decrease with time.~~

Thus, in accordance with Eq. (11) that was obtained from Eqs. (1)-(10) for $\mathrm{Bu} << 1$, $\mathrm{Ri} >> 1$ and $\mathrm{Pr} = 1$, the large-scale disturbances can be unstable. Such instability has to be distinguished from the diffusive instability (McIntyre, 1970; Baker, 1970; Calman, 1976) which occurs when $\mathrm{Ri} < (\mathrm{Pr}+1)^2/4\,\mathrm{Pr}$ and is absent at $\mathrm{Pr} = 1$. One of the important distinctions between these two models of baroclinic front instability is that in the present model the disturbances are allowed to have a nonzero slope in the along-front direction while in the model of diffusive instability by McIntyre (1970) the slope is taken to be zero. Therefore, the McIntyre's model and other models in which the term $\partial p/\partial x$, where x is the along-front co-ordinate, in the equations of motions is ignored (McIntyre, 1970; Baker, 1971; Calman, 1977; Munro et al., 2010), can be referred to as the 2D models.

From the mathematical point of view, the models that take into account the along-front slope of disturbances, are much more complicated. Indeed, the analysis of instability in the 2D models ultimately reduces to finding the roots of a polynomial depending upon the wave-numbers and growth rate. The models, that take into account the along-front slope of disturbances, are reduced to the differential equations with variable, z-dependent coefficients, and such problems can be solved analytically only in rare cases.

Also, it is worth to note that the instability described by Eq. (11) is not the critical layer instability analyzed by Miles (Miles, 1965) for a geostrophic current with constant vertical shear based on similarity between the equation of potential vorticity and the Orr-Sommerfeld equation. Indeed, the phase velocity of the unstable disturbances in our model satisfies the inequality $U_1 + U_3 - \mathrm{Re}\, c \neq 0$ for any value of the vertical coordinate *z*. As we can see from Eq. (16), the phase velocity of unstable disturbances is directed along the geostrophic current and exceeds the maximum velocity of the current. In such a case, in the author's opinion, the most likely is the conversion of the kinetic energy of the main flow into the kinetic energy of disturbances.

2.2 Stable solutions

Stable solutions of Eq. (13) are realized ~~at~~ for $\mathrm{Re}\, a > 0$. In this case $F_1(z)$ and all its derivatives vanish ~~at~~ for $z \to \pm\infty$, but $F_3(z)$ and all its derivatives increase infinitely for ~~at~~ $z \to \pm\infty$. To construct our own functions of the eigenvalue problem for the case 1 ($2H_0 \sim H << fL/N$), we have to take $B_3 = 0$.

The solutions describe slow time-decay~~ing~~, long waves that can move, in contrast to the Rossby waves, not only to the west but also to the east depending on ~~according to~~ the sign of s (see Eq. (7)). Moreover, if $|s| > \beta N_0^2 / f^2$ (which is quite possible especially in polar regions), the long-wave dynamics in the β-plane approximation is determined by the linear shear of geostrophic flow rather than the β-effect.

The real and imaginary parts of the growth rate of stable perturbations are

$$kc_1 = -2.5 \cdot \sqrt{|s| kK} + kU_3,\ kc_2 = -2.5 \cdot \sqrt{|s| kK} \text{ ~~at~~ for } s < 0, \tag{19 a}$$

$$kc_1 = 2.5\sqrt{skK} + kU_3 \ \ kc_2 = -2.5 \cdot \sqrt{skK} \text{ ~~at~~ for } s > 0. \tag{19 b}$$

According to Eq. ~~In accordance with~~ (19), the condition $U_1 + U_3 - \mathrm{Re}\, c \neq 0$ is satisfied, if $2.5\sqrt{|s| kK / k} > |s| H_0^{\,2} / 2$. Comparing Eqs. (16) and (19), we can conclude, that the phase velocity has different sign for ~~of~~ the stable and unstable disturbances ~~has a different sign~~. That is, stable and unstable perturbations described by solutions Eq. (15) will move in opposite ~~different~~ directions with respect to the flow and a fixed observer.

For the case 1 ~~and~~ type 2 boundary conditions, a plot of the density disturbances ~~stable solutions corresponding to the disturbances of density~~ $\mathrm{Re}\,\rho = \mathrm{Re}(dF/dz) = \tilde{\rho}$ corresponding to the stable solutions is presented in Fig. 3.

2.3 Obtained solutions: ~~some comments~~ discussion

~~Our own functions, obtained in the previous subsections, have the vertical structure of the unstable perturbations, which differs significantly from those of the classical 3D and 2D interleaving models for double diffusive interleaving at the oceanic front (Stern, 1967; Toole and Georgi, 1981; McDougall, 1985a, 1985b; Niino, 1986; Yoshida et al., 1989; Kuzmina and Rodionov, 1992; May and Kelley, 1997; Kuzmina, 2000; Kuzmina and Zhurbas, 2000). Given that the intrusions in the oceanic fronts have different forms, the present results may be useful for interpreting empirical data. However, the simplicity of o~~

Our model does not allow ~~us~~ to determine the maximum growth rate. Here again we can see an analogy with the work by Stern (1967). Indeed, in a well-known paper by Stern (1967), which was the first study of the double diffusion instability of the infinite thermohaline front, the magnitude of the fastest growing mode was not found. The reason is that the growth rate in Stern's model could indefinitely increase with the horizontal wave number due to the neglected~~ of~~ vertical friction. A similar feature is typical for our model. ~~of a simple model described above.~~ The growth rate increases with the increase in the wave number k up to the limit $\tilde{k}$ for which the constraint of $\tilde{k} << f/(2H_0N_0)$ is still valid. Nevertheless, for a rough estimate of the time of formation of unstable perturbations it is reasonable to use ~~formula~~ Eq. (16). It is also worth evaluating the relationship between the growth rate of unstable disturbances and the layer thickness (case 1) or the characteristic vertical scale of disturbances (case 2). Let us address Eq. (17), which follows from the boundary conditions for one of the problems of ~~studying the~~ instability in a finite layer. The parameter $\chi = \mathrm{Re}(-aH_0^{\ 2}) = 0.5(ks/K)^{1/2}H_0^{\ 2}$ governs Eq. (17). The higher the value of χ ~~this parameter~~, the ~~greater~~ larger the wave number of the unstable mode for the given values ~~parameters~~ of the problem parameters~~,~~ K, s, and H_0, and therefore, the larger ~~greater~~ the growth rate. However, the applicability of our model imposes a constraint on the space of wave numbers, $k << f/(2N_0H_0)$. In order to satisfy these two conditions simultaneously in the wide range of variability of hydrological parameters in the ocean, it is reasonable to put $1 \le \chi \le 2$. For $\chi = 2$, taking into account Eq. (16), we obtain the following formula relating the growth rate of disturbances and the vertical scale of the layer: $kc_2 = 1/T = 10 \cdot K/H_0^{\ 2}$.

~~For damped with increasing $|z|$ solutions $F_3(z)$, the length scale $H = 2(K/ks)^{1/4}$ determines the characteristic vertical scale of the disturbances. Therefore, when $\chi = 2$ the characteristic vertical scale of disturbances is a scale on which the perturbation amplitude decreases by a factor $e = 2.718...$. The formula relating the growth rate and the vertical scale of the disturbances will be the same as the previous one, but with H instead of H_0.~~

It is easy to understand the physical meaning of the parameter χ. This parameter characterizes the ratio of advection and vertical diffusion terms depending on the wave number k. Indeed, recalling ~~if we take into account~~ that in our model $U = U_1 + U_3$ and ~~take zero~~ the geostrophic velocity on the boundaries of the layer is zero, ~~$(U_3 = -sH_0^{\ 2}/2,\ s>0)$,~~ the maximum velocity at the midline of the layer ~~will~~ shall be $U_{\max} = |s|H_0^{\ 2}/2$. This allows the squared parameter χ to be presented as $\chi^2 = 0.25(ks/K)H_0^{\ 4} = 0.5 \cdot R_d k H_0$, where $R_d = 0.5sH_0^{\ 3}/K = U_{\max}H_0/K$ is a diffusion analogue of the Reynolds number called the ~~or~~ Peclet number.

~~It is worth noting that Eq. (13), being differentiated, corresponds to a simplified form of the Orr–Sommerfeld equation (see e.g. (Lin, 1955; Stern, 1975)) written under the extra-long-wave approximation. However, there are a number of differences between these equations, namely: (a) the destabilising factor in the Orr–Sommerfeld equation is friction, rather than diffusion, (b) the unknown function is a stream function in the vertical plane, and, finally, (c) to analyse instability in the frame of the Orr–Sommerfeld equation it is suggested that viscosity is vanishingly small but finite. For this reason, the disturbances out of the crirical layer are described by the equation of an ideal fluid, but in the region of a thin critical layer the equation of the so-called "viscous regime" is used (see, for example, Miles, 1965).~~

~~To obtain the solutions for the "viscous regime", the Orr–Sommerfeld equation is greatly simplified: only the terms describing derivatives of the unknown function of the 4th and 2nd order are considered (Iordanskiy and Kulikovskiy, 1966). Moreover, the velocity of the flow $U(z)$ is linearised as $U(z) \approx U'_z(z_0)(z - z_0)$, where z_0 is a point lying on the midline of the critical layer. In this regard, the solutions obtained in the present work for a parabolic type of geostrophic flow are different from the solutions of the "viscous regime" of the Orr–Sommerfeld equation, which are expressed by Hankel functions of order 1/3 (Lin, 1955). The unstable modes described by solutions (15) cannot be attributed to the instability of the critical layer: see~~

~~formulas (16) describing the phase velocity of unstable modes. Thus, there are significant differences between the approaches used to study instability in the frame of the Orr–Sommerfeld equation [see also the paper by Miles (Miles, 1965)] and the approach proposed in the present work.~~

To conclude this section, we note the following.

The instability of the weak geostrophic flow in the frame of the solutions Eq. (15) is an oscillatory instability (the growth rate has real and imaginary components). Generally, using interleaving models (Stern, 1967; Tool and Georgy, 1981; McDougal, 1985a, 1985b; Niino, 1986; Yoshida et al., 1989; Kuzmina and Rodionov, 1992; May and Kelley, 1997; Kuzmina and Zhurbas, 2000; Walsh and Ruddick, 2000; Merryfield, 2002), it is possible to obtain the monotonous unstable modes only (the phase velocity of the disturbances is equal to zero: $\mathrm{Re}\,c = 0$ ~~.~~). The exceptions to this rule are the interleaving models related to ~~describing the interleaving in the~~ equatorial fronts. In accordance with the modelling efforts (Richards, 1991; Edwards and Richards, 1999; Kuzmina et al., 2004; Kuzmina and Lee, 2005), the instability of the equatorial fronts in the scale of intrusive layering is regarded as an oscillatory instability.

~~The g~~General solution Eq. (15) is one of the classes of solutions of Eq. (13). Thus, for example, at $ikc = 9a \cdot K + ikU_3$ it is also possible to find an analytical~~ly~~ general solution of (13) ~~too~~. This solution ~~will~~ would have a more complex structure than the solution Eq. (15). The ~~D~~detailed analytical consideration of unstable modes based on the analysis of different classes of solutions of Eq. (13) taking into account the friction may be a subject for further research. In order ~~T~~to clearly define the range of applicability of our model, it would be ~~is~~ worth solving the eigenvalue problem for Eq. (7) for small values of parameter δ^2 by means of numerical methods. This problem also may be a ~~the~~ subject for ~~of~~ further research ~~too~~. The analytical solutions found can be used to validate numerical solutions of the eigenvalue problems. Moreover, the analytical solutions obtained provide ~~give~~ analytical expressions ~~formulas~~ for ~~own~~ eigenfunctions, phase velocities and growth/decay rates of disturbances that cannot, as a rule, be found exactly from numerical solutions.

3 Application to thermohaline intrusions in the Eurasian Basin of the Arctic Ocean

It is worth evaluating the time of formation of ~~the~~ large-scale intrusions based on the results of the presented model. According to Kuzmina et al. (2011), in the upper layer of the Polar Deep Water (PDW) where the large-scale intrusions are observed in the Eurasian Basin at stable–stable stratification, the following estimates of N, f, and β are typical: $N \approx 2 \cdot 10^{-3}$ s^{-1}, $f = 1.4 \cdot 10^{-4}$ s^{-1}, and $\beta < 0.3 \cdot 10^{-11}$ m^{-1}s^{-1} (at latitude of 83ºN and higher). Therefore, for disturbances, for example, with the vertical scale of $h = 100$ m, the Rossby radius of deformation is only $hN/f \approx 1$ km.

According to the derivation of Eq. (7), the value of the linear shear s is limited by the inequality of $|s| fL << N_0{}^2$. Given that the horizontal scale of the baroclinic fronts (along the cross-front axis y) in the upper layer of the PDW is approximately $L \approx 50$–100 km [see examples of transections across the fronts of different types observed in the PDW (Kuzmina et al., 2011)], the maximal linear shear can be estimated as $|s| \approx (1\text{–}2) \cdot 10^{-7}$ m^{-1}s^{-1}. Such value of the linear shear is large enough to neglect the β-effect term relative to the linear shear term in Eq. (7): $\beta N_0^2 / f^2 s < 0.6 \cdot 10^{-2} << 1$. The ~~V~~vertical diffusivity K can be estimated in the range of $K =$ (1–3)~~10^{-6}–3~~$\cdot 10^{-6}$ m^{2}s^{-1} (Merryfield, 2002; Walsh and Carmack, 2003). ~~[see also the following paper where the evaluations of coefficients of diffusivity in the Arctic thermocline were considered (Walsh and Carmack, 2003)].~~ We suggest a weak turbulence regime in the layer under consideration: $\Pr > 1$, $\Pr \cdot \mathrm{Bu} << 1$. The typical vertical scale of intrusive layering in the fronts of PDW is approximately 30–40 m (Merryfield, 2002; Kuzmina et al., 2014). Let us evaluate the time of formation of intrusions with the vertical scale of ~ 40 m. Using ~~the following~~ formula $k_0 c = 10 \cdot K / H_0{}^2$ (see Subsection 2.3), ~~$k_0 c = 10 \cdot K / H_0{}^2$,~~ we can estimate the time of formation ~~obtain that the time formation~~ of the unstable mode ~~is estimated~~ as $1/(k_0 c_2) \sim 5$ years for ~~at~~ $K = 10^{-6}$ m^{2}s^{-1} and approximately 2 years for ~~at~~ $K = 3 \cdot 10^{-6}$ m^{2}s^{-1}.

To verify the applicability of our model, it is worth ~~necessary~~ to estimate the wave number k_0 ~~,~~ using the following formula (see Subsection 2.3):

$$k_0 = 16 \cdot K / (H_0{}^4 s) . \quad (20)$$

Substituting $H_0 = 40$ m, $s = 2 \cdot 10^{-7}$ m^{-1}s^{-1}, and $K = 10^{-6}$ m^{2}s^{-1} in Eq. (20), we find ~~$k_0 = 0.2 \cdot 10^{-3}$~~ $k_0 = 3 \cdot 10^{-5}$ m^{-1}. The value of $k_0 \approx 10^{-4}$ m^{-1} may be obtained at $s = 2 \cdot 10^{-7}$ m^{-1}s^{-1}, $K = 3 \cdot 10^{-6}$ m^{2}s^{-1}. These ~~is~~ values of k_0 lies in the wave number range of applicability of our model, since $\delta^2 = N_0{}^2 H^2 k_0^2 / f^2 \sim 10^{-4} - 10^{-3}$ ~~$\delta^2 \approx 4 \cdot 10^{-3} - 4 \cdot 10^{-4}$~~.

The above-presented estimates of the time of formation ~~time~~ of intrusions in PDW are evidently better than the evaluations ~~that can be obtained~~ from 2D modelling of baroclinic front instability (see Introduction).

In the closing of this section, let us justify the assumption that the circulations associated with changes in vorticity Δp are not essential in the description of the formation of intrusions in all considered ~~the~~ cases ~~considered~~. According to Eqs. (2) and (10), the characteristic scale of vertical velocity in such circulations can be written as $w_1 \sim U \cdot u \cdot H \cdot k_0^2 / f$. In all the considered above ~~considered~~ cases of ~~the~~ application of the model to the Arctic intrusions, the relation of $U \cdot k_0 < 10^{-8}$ s^{-1} ~~following ratio~~ is satisfied. ~~$U / \tilde{L} < 10^{-8}$ s^{-1}.~~ Given that small disturbances of horizontal velocity cannot exceed the value of geostrophic velocity U, we find $w_1 < 4 \cdot 10^{-11}$ m s^{-1}. A liquid particle with such vertical velocity travels less than 0.004 m over ~~for~~ the period of formation of intrusion ~~formation~~ ($t_* = 1/(k_0 c_2) \approx 3$ years), while due to the vertical diffusion, the particle displacement is estimated as $\sqrt{K \cdot t_*} \approx$ 40 m (i.e. four orders of magnitude larger). ~~fluid particle can cover a distance of 40 m in approximately the same period.~~ Note also, that the decreasing with ~~damped with increasing~~ $|z|$ solution $F_3(z)$ can be used for the description of generation of intrusions ~~generation~~ even if the vertical velocity is not negligibly small ~~e quantity~~.

4 Conclusions

In this paper, we investigated analytically the ~~3D~~ instability of a baroclinic front in the quasi-geostrophic, long-wave approximation taking into account the vertical diffusion of buoyancy. ~~It is shown for the first time that not only double diffusion, but the diffusion of buoyancy can cause destabilization of the geostrophic flow.~~ Such instability has to be distinguished from the 2D McIntyre instability (McIntyre, 1970), the ~~a type of~~ instability due to flow-dependent fluctuations in turbulent diffusivities (Smyth and Ruddick, 2010), and the 2D baroclinic instability due to the double diffusion (Kuzmina and Rodionov, 1992; May and Kelley, 1997; Kuzmina and Zhurbas, 2000; Kuzmina and Lee, 2005).

In contrast to the ~~work~~ paper by Miles (Miles, 1965), who showed, that the ~~in which it was shown that the influence of~~ vertical diffusion of buoyancy is not essential in comparison with the ~~influence of~~ vorticity change in the destabilisation of ~~to destabilize the~~ zonal flow, we considered the opposite case, ~~when~~ where the vertical diffusion of buoyancy can play an important role as a destabilizer~~ation factor~~ of a very weak geostrophic current with linear shear and large cross-frontal scale.

The model we developed can be considered as a modification of Stern's model (Stern, 1967). However, instead of analysing the instability of a purely thermohaline front due to the double diffusion (Stern, 1967), in our case the instability of a weak baroclinic front is analysed taking into account the vertical diffusion of density. This model can be useful for describing stable and unstable disturbances of the ~~a~~ planetary scale in the polar regions of the ocean under the stable–stable stratification, particularly in the deep water of the Arctic Ocean~~Basin~~, where weak baroclinic fronts with a large horizontal (cross-frontal) scale and typical temporal variability period of the order of several years or more ~~are~~ have been observed, and the ~~beta~~ β-effect is negligible.

The stable ~~(decaying with time)~~ solutions are shown to describe long-wave disturbances ~~that~~, which, unlike Rossby waves, can move not only to the west but also to the east, depending on the magnitude and sign of the linear shear of geostrophic velocity. It is important to underline, that the linear shear of the mean flow (parabolic z-dependence of the mean velocity) affects ~~upon vertical co-ordinate) has an action upon~~ the dynamics of disturbances ~~and~~ likewise the β-effect.

The ~~U~~unstable ~~(increasing with time)~~ solutions can contribute to better understanding of ~~are used to describe~~ the formation of large-scale intrusions at ~~in the areas of~~ baroclinic fronts of ~~, which are observed in~~ the Arctic Ocean ~~Basin~~ in the layers ~~regions~~ characterized by ~~an~~ absolutely stable thermohaline stratification, for example, in the upper layer of the PDW in the Eurasian Basin. It is important, that the vertical scale of the new modes of instability can reach tens of meters of magnitude, just in accordance with the observations. However, the model is so complex that obtaining the comprehensive results of modelling that can be fully comparable with the empirical data, would still remain a future task.

~~The proposed description of intrusions generation in baroclinic fronts can be considered as a possible alternative mechanism relative to the differential mixing. However, at the moment this is just a hypothesis, and further efforts, both in theoretical modelling and field data analysis, are needed to justify it.~~

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Figures

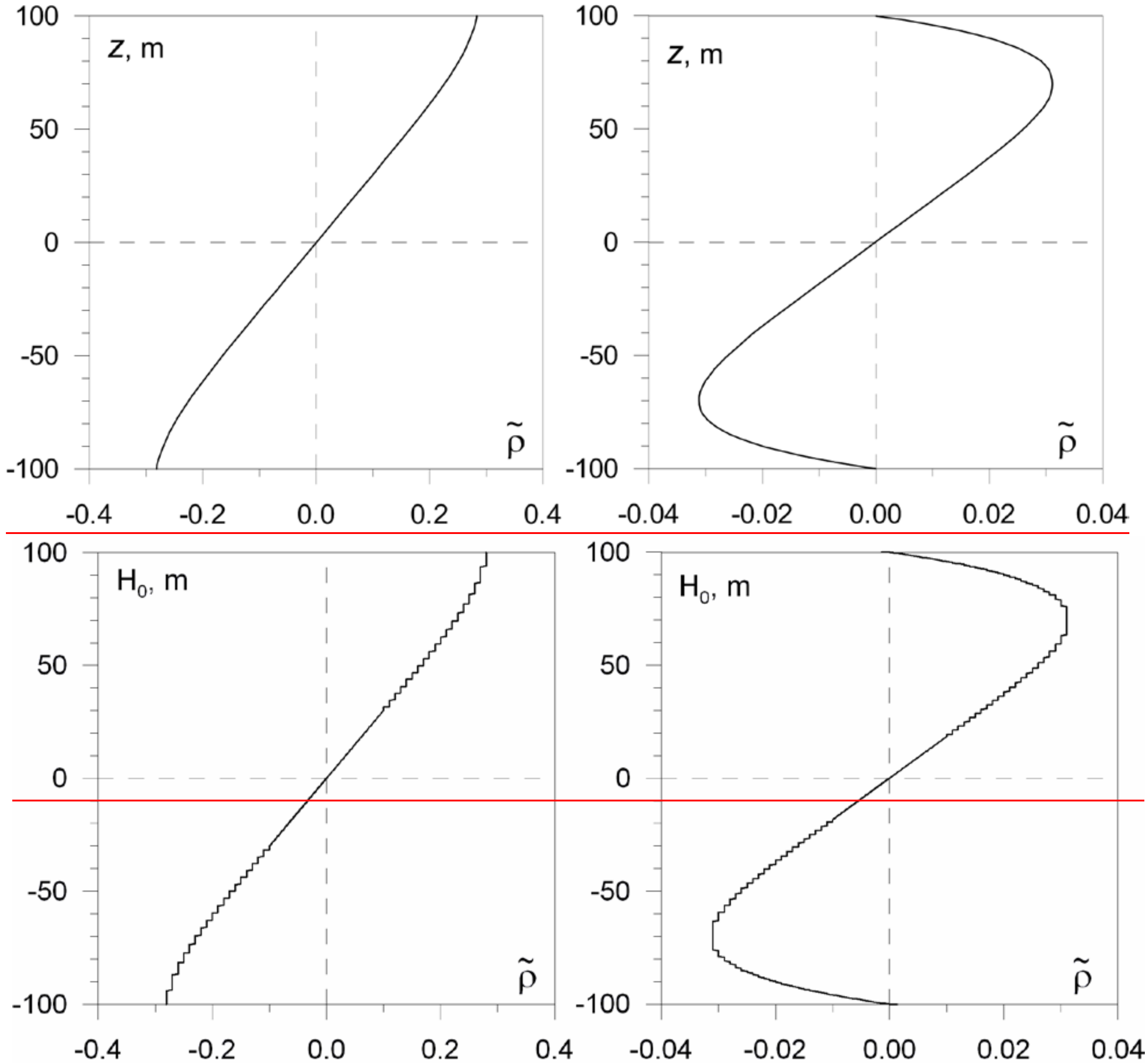

Figure 1. Modelled vertical profiles of density disturbances $\mathrm{Re}(dF/dz) = \mathrm{Re}\,\rho = \tilde{\rho}$ **for case 1. Unstable (growing) solution for boundary conditions of type 1 (left) and type 2 (right) for ~~and value of~~** $\chi = \mathrm{Re}(-aH_0^{\ 2}) = 0.5(ks/K)^{1/2} H_0^{\ 2} = 1.5$**,** $K = 10^{-5}$ **m**^{2} **s**$^{-1}$**,** $H_0 = 100$ **m,** $s = 10^{-7}$ **m**$^{-1}$**s**$^{-1}$**.**

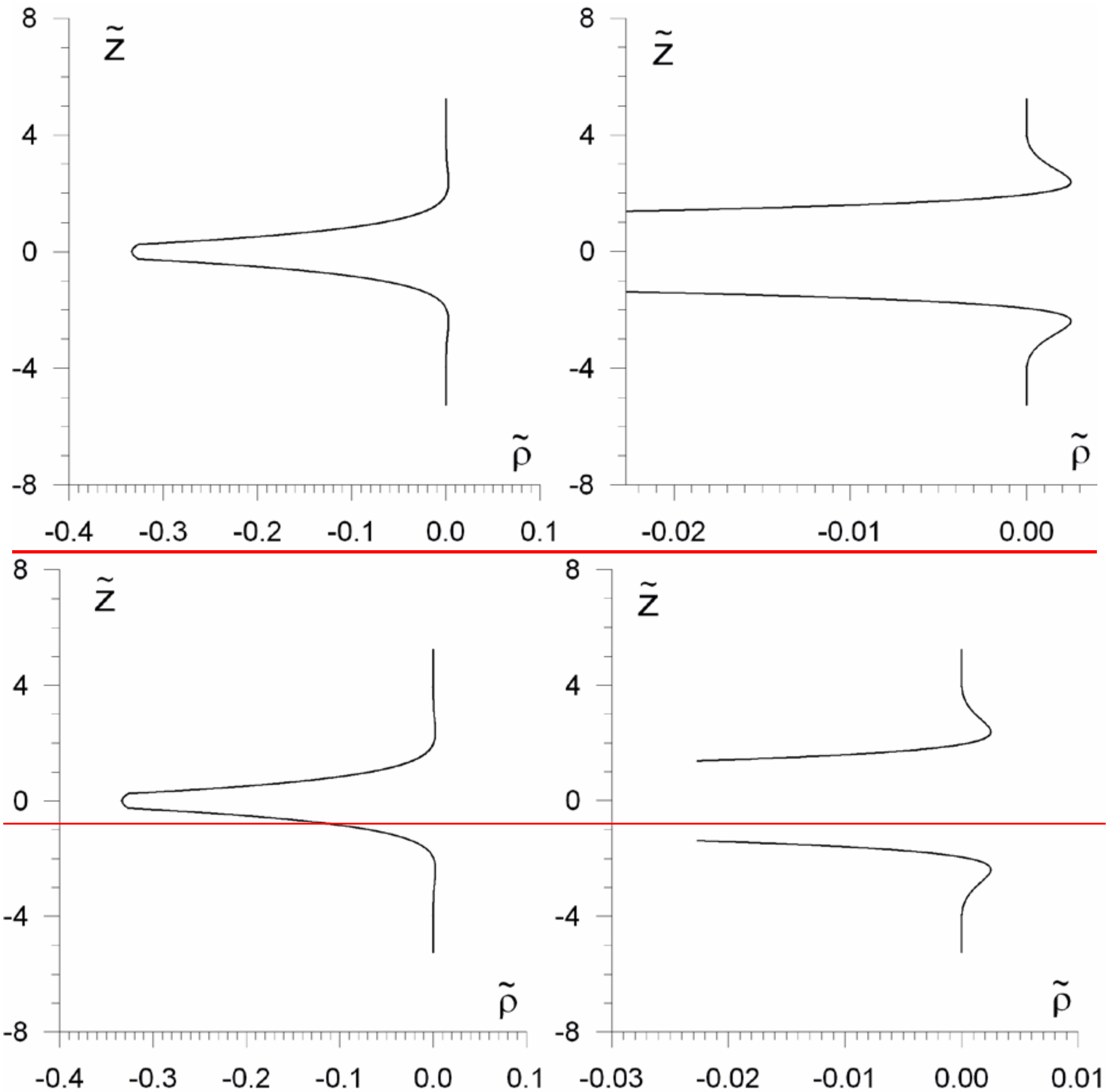

Figure 2. Modelled vertical profile of density disturbances $\mathrm{Re}(dF/dz) = \mathrm{Re}\,\rho = \tilde{\rho}$ **for case 2. The function** $\mathrm{Re}(dF_3/dz)$ **(left) and its stretched fragment ~~of this function~~ (right) versus the ~~depending on~~ dimensionless co-ordinate** $\tilde{z} = z \cdot (ks/K)^{1/4}$ **are presented for** $k = 10^{-5}$ **m^{-1},** $s = 10^{-7}$ **m^{-1}s^{-1},** $K = 10^{-5}$ **m^{2}s^{-1}.**

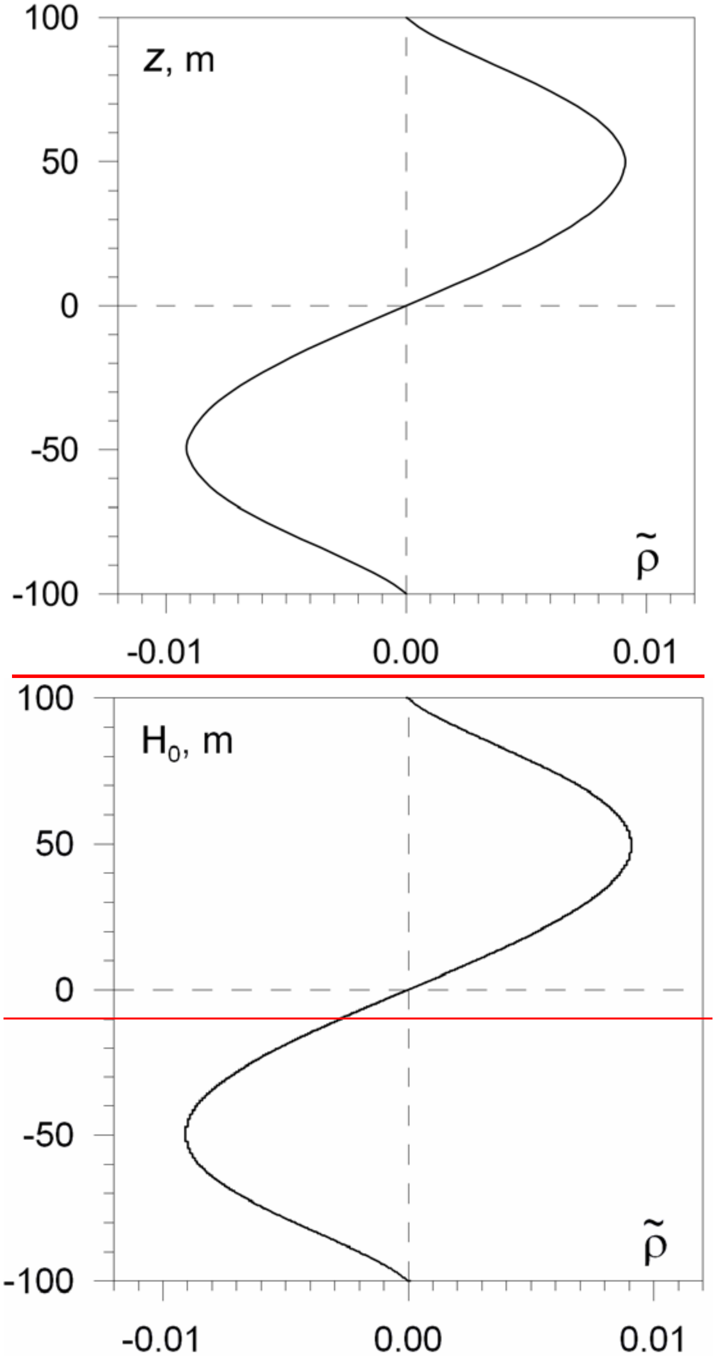

Figure 3. Stable solution for case 1 and boundary conditions of type 2 for ~~at~~ $\chi = \mathrm{Re}(-aH_0^{\,2}) = 0.5(ks/K)^{1/2} H_0^{\,2} = 2$**,** $K = 10^{-5}$ **m**^{2} **s**$^{-1}$**,** $H_0 = 100$ **m,** $s = 10^{-7}$ **m**$^{-1}$**s**$^{-1}$**.**