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Understanding mixing efficiency in the oceans: Do the nonlinearities of the equation of state for seawater matter?

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Abstract. There exist two central measures of turbulent mixing in turbulent stratified fluids that are both caused by molecular diffusion: 1) the dissipation rate $D(APE)$ of available potential energy APE; 2) the turbulent rate of change $W_{r,turbulent}$ of background gravitational potential energy GPE_r. So far, these two quantities have often been regarded as the same energy conversion, namely the irreversible conversion of APE into GPE_r, owing to the well known exact equality $D(APE) = W_{r,turbulent}$ for a Boussinesq fluid with a linear equation of state. Recently, however, Tailleux (2009) pointed out that the above equality no longer holds for a thermally-stratified compressible, with the ratio $\xi = W_{r,turbulent}/D(APE)$ being generally lower than unity and sometimes even negative for water or seawater, and argued that $D(APE)$ and $W_{r,turbulent}$ actually represent two distinct types of energy conversion, respectively the dissipation of APE into one particular subcomponent of internal energy called the 'dead' internal energy IE_0, and the conversion between GPE_r and a different subcomponent of internal energy called 'exergy' IE_{exergy}. In this paper, the behaviour of the ratio ξ is examined for different stratifications having all the same buoyancy frequency N vertical profile, but different vertical profiles of the parameter $\Upsilon = \alpha P/(\rho C_p)$, where α is the thermal expansion coefficient, P the hydrostatic pressure, ρ the density, and C_p the specific heat capacity at constant pressure, the equation of state being that for seawater for different particular constant values of salinity. It is found that ξ and $W_{r,turbulent}$ depend critically on the sign and magnitude of $d\Upsilon/dz$, in contrast with $D(APE)$, which appears largely unaffected by the latter. These results have important consequences for how the mixing efficiency should be defined and measured in practice, which are discussed.

1 Introduction

As is well known, turbulent diffusive mixing in the oceans is a physical process that it is crucially important to parameterise correctly in numerical ocean models. Indeed, it is the quality of the parameterised irreversible diabatic processes in such models that largely determine the realism of the simulated distribution of water mass properties, as well as of the behaviour of the so-called meridional overturning circulation and its associated heat transport (Gregg , 1987), which are two essential components of the large-scale ocean circulation that may interact with Earth climate. For this reason, much effort has been devoted over the past decades toward understanding the fundamental characteristics of turbulent diffusive mixing in stratified fluids, one important goal being to design physically-based parameterisations of irreversible mixing processes suitable for implementation in numerical ocean models used for climate change simulations.

At a fundamental level, turbulent diffusive mixing in stratified fluids is important for at least two distinct — although inter-related — reasons: 1) for its role as a mechanism responsible for a significant fraction — called the mixing efficiency — of the total irreversible decay of available mechanical energy (i.e., the sum of the kinetic energy KE and available potential energy APE); for its role as a mechanism responsible for the diffusive mixing of temperature across isopycnal surfaces, called diapycnal mixing. In the oceans, turbulent diapycnal mixing is essential to transfer the heat from the surface at a sufficiently rapid rate to balance the cooling of the deep ocean by high-latitude cooling and its associated deep water formation. In the turbulence literature, these two distinct roles of turbulent diffusive mixing are often being regarded as being associated with a single physical process, whereby the diffusively dissipated available potential energy is converted irreversibly into the background gravitational potential energy GPE_r, e.g., Winters & al. (1995); Peltier & Caulfield (2003). Although there exists a consider-

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able literature about how one should parameterise turbulent diffusive mixing, the model that appears the most often used or cited appears to be that of Osborn (1980), viz.,

$$K_\rho = \frac{\gamma_{mixing}\varepsilon}{N^2}, \tag{1}$$

where the turbulent diapycnal diffusivity K_ρ is expressed in terms of the kinetic energy dissipation ε, the so-called mixing efficiency γ_{mixing}, and squared buoyancy frequency N^2.

To the extent that γ_{mixing} can be regarded as some kind of universal parameter, as is often assumed [1], then Eq. (1) can be interpreted as stating that the amount of irreversible diffusive mixing is proportional to the sources of mechanical stirring. In the ocean modelling practice, however, the value of K_ρ is usually adjusted in order to reproduce the observed features of the oceanic stratification, which controls the strength of the simulated heat transport in ocean models. Such an approach was pioneered by Munk (1966), who sought to estimate K_ρ by assuming the stratification to obey the vertical advective/diffusive balance:

$$w\frac{\partial\theta}{\partial z} = \frac{\partial}{\partial z}\left(K_\rho\frac{\partial\theta}{\partial z}\right), \tag{2}$$

where θ is the potential temperature, which led to the widespread idea that the canonical value $K_\rho = 10^{-4}\,\mathrm{m}^2/\mathrm{s}$ was apparently needed to explain the observed structure of the oeanic thermocline, and consequently for achieving a meridional heat transport of the observed strength. Physically, the vertical advection/diffusion equation states that the upward advection of deep cold water is balanced by the downward turbulent diffusion of heat, where the upwelling velocity is assumed to be set by the rate of deep water formation. Such an approach, however, is indirect, and does not address the issue of whether there is enough mechanical energy in the oceans to maintain the stirring required to achieve a value of turbulent diapycnal mixing about three orders of magnitude larger than the molecular diffusivity of heat, an issue that was only undertaken by Munk & Wunsch (1998) about a decade ago, as discussed in further details below. Whether this is the case was questioned in subsequent years, with several observational studies suggesting that K_ρ in the oceans interior was in general typically smaller by an order of magnitude than Munk (1966)'s value e.g., see Ledwell & al (1998) and the review by Gregg (1987), prompting much debate in the ocean community as to correctness and accuracy of Munk (1966)'s estimate. On the other hand, it is also widely recognised that turbulent mixing in the oceans is highly variable, both spatially and temporally, and therefore certainly not well described by a single value independent of space and time. For that reason, Munk and Wunsch (1998) suggested to resolve that dilemma by regarding the value $K_\rho = 10^{-4}\,\mathrm{m}^2/\mathrm{s}$ as a bulk-averaged value to be interpreted as resulting from the overall effect of weak interior values combined with intense turbulent mixing occurring in coastal areas or over rough topography. In a second step, they also proposed to use Osborn (1980)'s model to see whether there is enough mechanical energy to support stirring in the oceans.

Recently, these ideas have been used to examine whether the energetics of turbulent irreversible diffusive mixing impose constraints on the magnitude of the mechanical sources of stirring. To investigate this issue, Munk & Wunsch (1998) examined the budget of gravitational potential energy, which they argue must be a balance between the rate of GPE loss due to cooling and the rate of GPE increase due to turbulent diffusive mixing, i.e.,

$$\left|\frac{d(GPE)}{dt}\right|_{cooling} \approx \left|\frac{d(GPE)}{dt}\right|_{mixing}, \tag{3}$$

this result being obtained by multiplying Eq. (2) by $\alpha\rho_0 g z$, after some manipulation involving integration by parts and the neglect of surface heating, where α is the thermal expansion (assumed constant), g the acceleration of gravity, ρ_0 a reference density, and z the vertical coordinate pointing upward. Assuming the rate of GPE loss due to cooling to be known from an estimate of the North-Atlantic deep water formation rate, Munk & Wunsch (1998) invoke Osborn model to link the rate of GPE increase du to turbulent mixing to the work done by the mechanical sources of stirring $G(KE)$ by:

$$\left|\frac{d(GPE)}{dt}\right|_{mixing} \approx \gamma_{mixing} G(KE) \tag{4}$$

via the mixing efficiency γ_{mixing}. This approach, however, appears to rely on two important widespread and somewhat misleading ideas about turbulent mixing, namely: 1) that all of the kinetic energy eventually dissipated by diffusive turbulent mixing yields a corresponding increase in GPE; 2) that γ_{mixing} is some kind of universal parameter with a fixed value close to 0.2. Yet, it is clear from the work by Fofonoff (1998, 2001) that there is no guarantee in general that the background GPE should necessarily increase as the result of turbulent mixing, as this depends on the particular vertical temperature profile and nonlinear character of the equation of state considered. To be specific, whether the background GPE should increase or decrease as the result of turbulent mixing appears to depend on the sign of the following parameter:

$$\frac{d}{dz}\left(\frac{\alpha P}{\rho C_p}\right) \tag{5}$$

where α is the thermal expansion coefficient, P is the pressure, ρ is the density, and C_p is the specific heat capacity at constant pressure. Thus, the classical regime for which the background GPE increases as the result of turbulent irreversible mixing corresponds to the case where the sign of the above parameter is negative, as is the case if $\alpha/(\rho C_p)$ can

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[1] See Tailleux (2009), however, for physical arguments challenging this idea.

be regarded as nearly constant, assuming the pressure to be hydrostatic (so that $dP/dz = -\rho g < 0$). This is the case, in particular, for a Boussinesq fluid with a linear equation of state, for which the increase of GPE due to turbulent mixing is accurately described by the following formula:

$$\left\{ \frac{d(GPE)}{dt} \right\}_{mixing} = \int K_\rho \rho N^2 dV. \qquad (6)$$

In the literature, the idea according to which the background GPE should always increase as the result of turbulent mixing appears to be linked to the widespread belief that when the available potential energy (APE) is removed by molecular diffusion, it is necessarily irreversibly converted into background GPE, as proposed by Winters & al. (1995) for instance. Physically, this idea results from regarding the APE dissipation rate $D(APE)$ and the turbulent rate of GPE_r change $W_{r,turbulent}$ as being basically the same conversion, which is motivated by the fact that $D(APE) = W_{r,turbulent}$ for a Boussinesq fluid with a linear equation of state. This idea, however, was recently challenged by Tailleux (2009), who pointed out that the previous equality is actually a serendipitous feature of the Boussinesq approximation, but that $D(APE)$ and $W_{r,turbulent}$ can in fact be very different from each other for a real fluid, suggesting on the contrary that $D(APE)$ and $W_{r,turbulent}$ represent two distinct measures of turbulent irreversible diffusive mixing. Specifically, Tailleux (2009) argued that $D(APE)$ physically represents the dissipation rate of APE into a particular subcomponent of internal energy, called the 'dead internal energy' IE_0, while $W_{r,turbulent}$, like $W_{r,mixing}$, physically represents the conversion rate between GPE_r and a different subcomponent of internal energy, called the 'exergy' IE_{exergy}. Physically, the 'dead' and 'exergy' components of internal energy can be regarded as being associated respectively with the equivalent thermodynamic equilibrium temperature $T_0(t)$ and vertical temperature stratification $T_r(z,t)$ of the fluid. Physically, it means that the APE dissipation mainly results in increasing the equivalent thermodynamic temperature $T_0(t)$, whereas the conversion $W_{r,turbulent}$ associated with the GPE_r variations results in the smoothing out of the vertical temperature stratification $T_r(z,t)$.

Until now, most theoretical and numerical descriptions of turbulent mixing in stratified fluids have most often relied on the incompressible Navier-Stokes equations, often in the context of the Boussinesq approximation. With regard to the thermodynamics of the fluid, it is generally considered to be unimportant at leading order, and as a result, a linear equation of state is generally considered to be accurate enough for the purposes of describing turbulent mixing. As is well known, however, the equation of state for seawater — which is the one appropriate for describing the oceans — is strongly nonlinear in temperature, pressure, and salinity. The question arises, therefore, of which particular properties of turbulent mixing, if any, might be affected by the nonlinear nature of the equation of state for seawater. In order to examine this issue, this paper seeks to gain insights into how the ratio $\xi = D(APE)/W_{r,turbulent}$ is controlled by the nonlinearities of the equation of state, in the case where the fluid is water or seawater, and how this affects our understanding of the so-called mixing efficiency. Physically, mixing efficiency is often regarded as the fraction of the total mechanical stirring energy that is eventually dissipated by molecular diffusion, and is a central quantity in the study of turbulent diffusive mixing. Section 2 provides a theoretical formulation of the issue discussed. Section 3 discusses the methodology, while the results are presented in Section 4. Finally, section 5 summarises and discusses the results.

2 Theoretical formulation of the problem

2.1 Energetics of mixing

An important and longstanding issue in the study of turbulence has been to design satisfactory ways to cleanly separate the effects due to the adiabatic and reversible stirring process from those due to irreversible molecular diffusion. From a theoretical viewpoint, it is increasingly realized that the stirring and mixing processes are most easily distinguished if the potential energy (i.e., the sum of gravitational potential and internal energies) is partitioned into its "available" and "non-available" components, as initially proposed by Lorenz (1955) in the context of atmospheric energetics, and more recently by Winters & al. (1995) to study irreversible mixing in stratified turbulence. Physically, the underlying idea of the method is linked to the result that the probability density function (pdf in short) of the fluid parcels' entropy is only affected by molecular diffusion, but unaffected by the stirring process. From this, it follows that irreversible mixing effects can be neatly isolated by computing the time evolution of the pdf of entropy. As it turns out, the latter quantity coincides with the entropy distribution of the reference state entering the definition of Lorenz (1955)'s available potential energy. Such result is important, because it establishes a direct connection between the study of turbulent irreversible mixing and that of the energetics of stratified turbulence. In a Boussinesq fluid with a linear equation of state, the role of entropy is played by either temperature or density.

Following Winters & al. (1995) (using somewhat different notations), the use of Lorenz (1955)'s available potential energy to study irreversible mixing in the context of freely decaying turbulence in an insulated domain relies on describing the energetics of the fluid in terms of the volume-integrated kinetic energy (KE), available potential energy (APE), and background gravitational potential energy GPE_r as follows:

$$\frac{d(KE)}{dt} = -C(KE,APE) - D(KE), \qquad (7)$$

$$\frac{d(APE)}{dt} = C(KE,APE) - D(APE), \qquad (8)$$

$$\frac{d(GPE_r)}{dt} = W_{r,mixing} = W_{r,laminar} + W_{r,turbulent}. \quad (9)$$

The right-hand side of the above equations represent various energy conversion terms. In absence of any irreversible processes, only one energy conversion term would remain, namely $C(KE, APE)$, which physically represents the reversible conversion between KE and APE, often referred to as the buoyancy flux. All other energy conversion terms are associated with irreversible processes, with $D(KE)$ the viscous KE dissipation rate, $D(APE)$ the dissipation rate of APE due to molecular diffusion, and $W_{r,mixing}$ the rate of change of GPE_r due to molecular diffusion, which it is customary to separate into a background laminar and turbulent contribution. It is important to note that the above equations represent domain-averaged equations, not local formulations, which are characterized by the lack of lateral energy fluxes in and out of the domain. Such budgets, therefore, are well suited to understanding laboratory experiments of turbulent mixing. In the ocean interior, the question arises of the importance of the lateral fluxes of kinetic and available potential energy.

As discussed by Tailleux (2009), Eqs. (7- 9) provide a unifying way to describe the energetics of both the incompressible Boussinesq and compressible Navier-Stokes equations, by adapting the definitions of the energy reservoirs and energy conversion terms to the particular set of equations considered. Explicit expressions for $D(APE)$ and $W_{r,mixing}$ are given by Tailleux (2009) in the particular cases of: 1) a Boussinesq fluid with a linear and nonlinear equation of state in temperature; 2) for a compressible thermally-stratified fluid obeying the Navier-Stokes equations of state with a general equation of state depending on temperature and pressure. These expressions are recalled further below for case 2). While $W_{r,laminar}$ is well understood to be a conversion between IE and GPE_r, the nature of the energy conversions associated with $D(APE)$ and $W_{r,turbulent}$ is still a matter of debate. Currently, it is widely assumed that $D(APE)$ and $W_{r,turbulent}$ represent the same kind of energy conversion, namely the irreversible conversion of APE into GPE_r owing to the fact that for a Boussinesq fluid with a linear equation of state (referred to as the L-Boussinesq model hereafter), one has the exact equality $D(APE) = W_{r,turbulent}$. It was pointed out by Tailleux (2009) that this equality is a serendipitous artifact of the L-Boussinesq model, which does not hold for more accurate forms of the equations of motion. More generally, Tailleux (2009) found that the ratio $\xi = W_{r,turbulent}/D(APE)$ is not only systematically lower than unity for water or seawater, but can in fact also takes on negative values, as previously discussed by Fofonoff (1962, 1998, 2001) in a series of little known papers. In other words, the equality $D(APE) = W_{r,turbulent}$ is only a mathematical equality, not a physical equality, by defining a physical equality as a mathematical equality between two quantities that persists for the most accurate forms of the governing equations of motion. To clarify the issue, Tailleux (2009) sought to understand the links between $D(APE)$, $W_{r,mixing}$ and internal energy, by establishing the following equations:

$$\frac{d(IE_0)}{dt} \approx D(KE) + D(APE), \quad (10)$$

$$\frac{d(IE_{exergy})}{dt} \approx = -\underbrace{[W_{r,laminar} + W_{r,turbulent}]}_{W_{r,mixing}}, \quad (11)$$

which demonstrate that the viscously dissipated KE and diffusively dissipated APE both end up into the dead part of internal energy IE_0, whereas $W_{r,mixing}$ represent the conversion rate between GPE_r and the 'exergy' component of internal energy IE_{exergy}. A schematic energy flowchart illustrating the above points is provided in Fig. 1.

2.2 Efficiency of mixing and mixing efficiency

The available potential energy framework introduced by Winters & al. (1995) and extended by Tailleux (2009) greatly simplifies the theoretical discussion of the concept of mixing efficiency, which plays a central role in the study of turbulent mixing, but whose nature is little understood. To that end, it is useful to start with the evolution equation for the total “available” mechanical energy $ME = KE + APE$, obtained by summing the evolution equations for KE and APE, leading to:

$$\frac{d(ME)}{dt} = -[D(KE) + D(APE)]. \quad (12)$$

Eq. (12), along with Eq. (10), are very important, for they show that both viscous and diffusive processes contribute to the dissipation of ME into deal internal energy IE_0. From this viewpoint, understanding turbulent diapycnal mixing amounts to understanding what controls the ratio $\gamma_{mixing} = D(APE)/D(KE)$, that is, the fraction of the total available mechanical energy dissipated by molecular diffusion rather than by molecular viscosity. The amount of ME dissipated by molecular diffusion, i.e., $D(APE)$, is important, because it is directly related to the definition of turbulent diapycnal diffusivity, which one may write as

$$K_\rho = \frac{D(APE)}{N^2}, \quad (13)$$

e.g., Osborn and Cox (1972), where N^2 is the squared buoyancy frequency. From the viewpoint of turbulent diapycnal mixing, a turbulent mixing event will be regarded as “efficient” if the mixing efficiency $\gamma_{mixing} = D(APE)/D(KE)$ is high. The concept of mixing efficiency is important, because provided that the value of γ_{mixing} is known, it provides a way to estimate K_ρ from measurements of turbulent viscous dissipation rates from

$$K_\rho = \frac{\gamma_{mixing} D(KE)}{N^2}, \quad (14)$$

as proposed by Osborn (1980). This approach, however, is potentially misleading, because viscous and diffusive dissipation mechanisms are largely distinct and independent physically, except in very special circumstances. It is important to realize, indeed, that a turbulent inviscid fluid [2] with nonzero molecular diffusion would have zero viscous dissipation of kinetic energy, but would still exhibit finite diapycnal mixing, in apparent contradiction with Osborn (1980)'s formula (unless one accepts the possibility of having $\gamma_{mixing} = +\infty$). This is because at a fundamental level, Osborn (1980)'s formula merely represents the definition of mixing efficiency, rather than a causality relationship linking $D(APE)$ and $D(KE)$. This point seems to be overlooked in many observational studies that simply measure viscous dissipation rates, assuming that a straightforward inference about K_ρ can be made by using $\gamma_{mixing} = 0.2$ regardless of circumstances, which seems questionable given that the available evidence is that γ_{mixing} may vary significantly depending on circumstances.

The above definition of mixing efficiency $\gamma_{mixing} = D(APE)/D(KE)$ was introduced by Tailleux (2009), who referred to it as the "dissipation" mixing efficiency, as it based on dissipation quantities, which differs somewhat from the way mixing efficiency has been discussed so far. The link between the dissipation mixing efficiency and more traditional definitions of mixing efficiency can be clarified in the light of the above energy equations, by investigating the energy budget of a notional "turbulent mixing event", defined here as an episode of intense mixing followed and preceded by laminar conditions (i.e., characterised by very weak mixing), during which KE and APE undergo a net change change $\Delta KE < 0$ and $\Delta APE < 0$. As far as we understand the problem, most familiar definitions of mixing efficiency appear to implicitly assume $\Delta APE \approx 0$, as is the case for a turbulent mixing event developing from a unstable stratified shear flow for instance, e.g., Peltier & Caulfield (2003). This point can be further clarified by comparing the energetics of turbulent mixing events developing from the shear flow instability with that developing from the Rayleigh-Taylor instability, treated next, which by contrast can be regarded as having the idealised signature $\Delta KE \approx 0$ and $\Delta APE < 0$.

In the case of the stratified shear flow instability, assumed to be such that $\Delta KE < 0$ and $\Delta APE \approx 0$, integrating the above energy equations over the time interval over which the turbulent mixing event takes place [3] yields:

$$\Delta KE = -\overline{C(KE,APE)} - \overline{D(KE)}, \tag{15}$$

$$0 = \overline{C(KE,APE)} - \overline{D(APE)}, \tag{16}$$

$$\Delta GPE_r = \overline{W}_{r,mixing} = \overline{W}_{r,turbulent} + \overline{W}_{r,laminar}, \tag{17}$$

where the overbar denotes the time integral over the mixing event. For a Boussinesq fluid with a linear equation of state, Winters & al. (1995) showed that $\overline{D(APE)} = \overline{W}_{r,turbulent}$. If we combine the latter result with the APE budget (i.e., Eq. (16)), one sees that one has the triple equality:

$$\overline{C(KE,APE)} = \overline{D(APE)} = \overline{W}_{r,turbulent}. \tag{18}$$

The triple equality Eq. (18) suggests that any of the three quantities $\overline{C(KE,APE)}$, $\overline{D(APE)}$, or $\overline{W}_{r,turbulent}$ can a priori serve to measure "the fraction of the kinetic energy that appears as the potential energy of the stratification", which is the traditional definition of the flux Richardson number proposed by Linden (1979). Historically, the buoyancy flux $\overline{C(KE,APE)}$ is the one that was initially regarded as the natural quantity to use for that purpose in an overwhelming majority of past studies of turbulent mixing. As a result, most existing studies of turbulent mixing define the turbulent diapycnal diffusivity, mixing efficiency, and flux Richardson number in terms of the buoyancy flux as follows:

$$K_\rho^{flux} = \frac{\overline{C(KE,APE)}}{N^2}, \tag{19}$$

$$\gamma_{mixing}^{flux} = \frac{\overline{C(KE,APE)}}{N^2}, \tag{20}$$

$$R_f^{flux} = \frac{\overline{C(KE,APE)}}{\overline{C(KE,APE)} + \overline{D(KE)}}. \tag{21}$$

It is easily verified that the above equations are consistent with those considered by Osborn (1980) for instance. Physically, however, there are fundamental problems in using the buoyancy flux to quantify irreversible diffusive mixing, because as pointed out by Caulfield & Peltier (2000), Staquet (2000) and Peltier & Caulfield (2003), $C(KE,APE)$ represents a reversible energy conversion, which usually takes on both large positive and negative values before settling on its long term average $\overline{D(APE)}$. Moreover, as pointed out below, the buoyancy flux is only related to irreversible diffusive mixing only if $\Delta APE \approx 0$ holds to a good approximation, for otherwise, it becomes also related to the irreversible viscous dissipation rate as shown by the KE budget

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[2] Inviscid is to be understood here as meaning zero viscosity only. The point is made because some authors used the term to also mean zero diffusivity, e.g., Paparella & Young (2002).

[3] It is usually assumed that the time average should be short enough that the viscous dissipation of the mean flow can be neglected. Alternatively, one should try to separate the laminar from the turbulent viscous dissipation rate. The following derivations assume that the viscous dissipation is dominated by the dissipation of the turbulent kinetic energy rather than that of the mean flow.

(Eq. (15)). Eq. (18) makes it possible, however, to use either $\overline{D(APE)}$ or $\overline{W}_{r,turbulent}$ instead of $\overline{C(KE,APE)}$ in the definitions (19) and (20). For this reason, both Caulfield & Peltier (2000) and Staquet (2000) proposed to measure the efficiency of mixing based on $\overline{W}_{r,turbulent}$, i.e.,

$$K_{\rho}^{GPEr} = \frac{\overline{W}_{r,turbulent}}{N^2}, \tag{22}$$

$$\gamma_{mixing}^{GPEr} = \frac{\overline{W}_{r,turbulent}}{D(KE)}, \tag{23}$$

$$R_f^{GPE_r} = \frac{\overline{W}_{r,turbulent}}{\overline{W}_{r,turbulent} + \overline{D(KE)}}, \tag{24}$$

such a definition being motivated by Winters & al. (1995)'s interpretation that $D(APE)$ and $W_{r,turbulent}$ represent the same energy conversion whereby the diffusively dissipated APE is irreversibly converted into GPE_r. The parameter $R_f^{GPE_r}$ was called the "cumulative mixing efficiency" by Peltier & Caulfield (2003) and modified flux Richardson number by Staquet (2000). As argued in Tailleux (2009), it is $\overline{D(APE)}$, rather than $\overline{W}_{r,turbulent}$, that directly measures the amount of KE eventually dissipated by molecular diffusion via its conversion into APE, suggesting that the flux Richardson number should actually be defined as:

$$R_f^{DAPE} = \frac{\overline{D(APE)}}{\overline{D(KE)} + \overline{D(APE)}}. \tag{25}$$

While the above formula makes it clear that all above definitions of R_f are equivalent in the particular case considered, it is easily realized that they will in general yield different numbers if one relaxes the assumption $\Delta APE \approx 0$ in Eq. (16), as well as the assumption of a linear equation of state, yielding a ratio $\xi = W_{r,turbulent}/D(APE)$ that is generally lower than unity and sometimes even negative for water or seawater. For this reason, it is crucial to understand the physics of mixing efficiency at the most fundamental level. From the literature, it seems clear that most investigators's idea about the flux Richardson number is as a quantity comprised between 0 and 1. From that viewpoint, the dissipation flux Richardson number R_f^{DAPE} is the only quantity that satisfies this property under the most general circumstances, as cases can easily be constructed for which both $\overline{W}_{r,turbulent}$ and $\overline{C(KE,APE)}$ are negative. Indeed, cases for which $\xi < 0$ are described in this paper, whereas $\overline{C(KE,APE)}$ is easily shown to be negative in the case of a turbulent mixing event for which all mechanical energy is initially provided entirely in APE form. In that case, assuming $\Delta APE < 0$ and $\Delta KE \approx 0$ in the above energy budget equations yields:

$$\overline{C(KE,APE)} = \overline{D(APE)} + \Delta APE = -\overline{D(KE)}, \tag{26}$$

which shows that this time, $\overline{C(KE,APE)}$ directly measures the amount of viscously dissipated kinetic energy, rather than diapycnal mixing. The latter case is relevant to understand the energy budget of the Rayleigh-Taylor instability, see Dalziel & al (2008) for a recent discussion of the latter.

2.3 Link between $D(APE)$ and $W_{r,mixing}$

In order to help the reader understand or appreciate why the ratio $\xi = W_{r,turbulent}/D(APE)$ is generally lower than unity for water or seawater, and hence potentially significantly different from the predictions of the L-Boussinesq model, it is useful to examine the structure of $W_{r,mixing}$ and $D(APE)$ in more details. As shown by Tailleux (2009), the analytical formula for the latter quantities in a fully compressible thermally-stratified fluid are given by:

$$W_{r,mixing} = \int_V \frac{\alpha_r P_r}{\rho_r C_{pr}} \nabla \cdot (\kappa \rho C_p \nabla T)\, dV, \tag{27}$$

$$D(APE) = -\int_V \frac{T - T_r}{T} \nabla \cdot (\kappa \rho C_p \nabla T)\, dV, \tag{28}$$

where as before α is the thermal expansion coefficient, P is the pressure, C_p is the specific heat capacity at constant pressure, ρ is density, with the subscript r indicating that values have to be estimated in their reference state. The parameter $\Upsilon = \alpha P/(\rho C_p)$ plays an important role in the problem. Physically, it can be shown that in an isobaric process during which the enthalpy of the fluid parcel increases by dH, the parameter Υ represents the fraction of dH that is not converted into internal energy, i.e., the fraction going into work (and hence contributing ultimately to the overall net change in GPE_r). As a result, Υ plays the role of a Carnot-like thermodynamic efficiency. In Eq. (27), Υ_r denotes the value that Υ would have if the corresponding fluid parcel was displaced adiabatically to its reference position.

In order to compare these two quantities, we expand T as a Taylor series around $P = P_r$, viz.,

$$T = T_r + \Gamma_r (P - P_r) + \ldots \tag{29}$$

where $\Gamma_r = \alpha_r T_r/(\rho_r C_{pr})$ is the adiabatic lapse rate. At leading order, therefore, one may rewrite $D(APE)$ as follows:

$$\begin{aligned} D(APE) &= \int_V \frac{\alpha_r (P_r - P)}{\rho_r C_{pr}} \frac{T_r}{T} \nabla \cdot (\kappa \rho C_p \nabla T) dV + \ldots \\ &= W_{r,mixing} + \int_V \frac{(T_r - T)}{T} \frac{\alpha_r P_r}{\rho_r C_{pr}} \nabla \cdot (\kappa \rho C_p \nabla T)\, dV \\ &- \int_V \frac{\alpha_r T_r P}{\rho_r C_{pr} T} \nabla \cdot (\kappa \rho C_p \nabla T)\, dV + \cdots \end{aligned} \tag{30}$$

These formula shows that $D(APE)$ can be written as the sum of $W_{r,mixing}$ plus some corrective terms. One sees that the L-Boussinesq model's results derived by Winters & al. (1995) can be recovered in the limit $T \approx T_r$, $P \approx -\rho_0 g z$, $\alpha_r/(\rho_r C_{pr}) \approx \alpha_0/(\rho_0 C_{p0})$, $\rho C_p \approx \rho_0 C_{p0}$, where the subscript 0 refers to a constant reference Boussinesq value, yielding:

$$D(APE) \approx W_{r,mixing} - W_{r,laminar} = W_{r,turbulent}. \tag{31}$$

These results, therefore, demonstrate that the strong correlation between $D(APE)$ and $W_{r,mixing}$ originates in both terms depending on molecular diffusion in a related, but nevertheless distinct, way, the differences between the two quantities being minimal for a linear equation of state. The fact that the two terms are never exactly equal in a real fluid clearly refutes Winters & al. (1995)'s widespread interpretation that $D(APE)$ and $W_{r,turbulent}$ physically represents the same energy conversion whereby the diffusively dissipated APE is irreversibly converted into GPE_r. In reality, $D(APE)$ and $W_{r,turbulent}$ represent two distinct types of energy conversions that happen to be both controlled by stirring and molecular diffusion in related ways, which explains why they appear to be always strongly correlated, and even exactly equal in the idealised limit of the L-Boussinesq model. If one accepts the above point, then it should be clear that what is now required to make progress is the understanding of what controls the behaviour of the parameter ξ, since the knowledge of the latter is obviously crucial to make inferences about turbulent diapycnal mixing from measuring the net changes of GPE_r for instance. The purpose of the numerical simulations described next is to help gaining insights into what controls ξ.

3 Methodology

To get insights into how the equation of state of seawater affects turbulent mixing, we compared $D(APE)$ and $W_{r,turbulent}$ for a number of different stratifications having the same buoyancy frequency vertical profile N, but different vertical profiles with regard to the parameter $\alpha P/(\rho C_p)$, as illustrated in Fig. 2. The quantities $D(APE)$ and $W_{r,mixing}$ were estimated from Eqs. (27) and (28), while $W_{r,turbulent}$ was estimated from

$$W_{r,turbulent} = W_{r,mixing} - W_{r,laminar}, \tag{32}$$

where $W_{r,laminar}$ was obtained by taking $T = T_r$ in the expression for $W_{r,mixing}$. The quantities $D(APE)$ and $W_{r,turbulent}$ were estimated numerically for a two-dimensional square domain discretised equally in the horizontal and vertical direction. In total, 27 different stratifications were considered, all possessing the same squared buoyancy frequency N^2 illustrated in the left panel of Figure 2, but different mean temperature, salinity, and pressure resulting in different profiles for the $\alpha P/(\rho C_p)$ parameter illustrated in the right panel of Figure 2. In all cases considered, the pressure varied from P_{min} to $P_{max} = P_{min} + 10dbar$, with P_{min} taking the three values (0 dbar, 1000 dbar, 2000 dbar). In all cases, the salinity was assumed to be constant, and taking one of the three possible values S = (30 Psu, 35 psu, 40 psu). With regard to the temperature profile, it was determined by imposing the particular value $T_{max} = T(P_{min})$ at the top of the fluid, with all remaining values determined by inversion of the buoyancy frequency N^2 common to all profiles by an iterative method. The imposition of a fixed buoyancy profile N, salinity S, pressure range, and minimum temperature T_{min} was found to yield widely different top-bottom temperature differences $T(P_{min}) - T(P_{max})$, ranging from a few tenths of degrees to about 4 degrees C depending on the case considered, as seen in Fig. 3. In each case, the thermodynamic properties of the fluid were estimated from the Gibbs function of Feistel (2003). Specific details for the temperature, pressure, and salinity in each of the 27 experiments can be found in Table 1 along with other key quantities discussed below.

Numerically, the two-dimensional domain used to quantify $D(APE)$ and $W_{r,turbulent}$ was discretised into $N_{pi} \times N_{pj}$ points in the horizontal and vertical, with $N_{pi} = N_{pj} = 100$. Mass conserving coordinates were chosen in the vertical, and regular spatial Cartesian coordinate in the horizontal. For practical purposes, the vertical mass conserving coordinate can be regarded as standard height z, as the differences between the two types of coordinates were found to be insignificant in the present context, and thus chose $\Delta x = \Delta z$. In order to compute $D(APE)$ and $W_{r,turbulent}$ for turbulent conditions, we modelled the stirring process by randomly shuffling the fluid parcels adiabatically from resting initial conditions. Shuffling the parcels in such a way requires a certain amount of stirring energy, which is equal to the available potential energy APE of the randomly shuffled state.

4 Results

For each of the 27 particular reference stratifications considered, synthetic turbulent states were constructed by generating hundreds of random permutations of the fluid parcels, thus simulating the effect of adiabatic shuffling by the stirring process, in each case yielding a particular value of $D(APE)$, $W_{r,mixing}$, $W_{r,turbulent}$ and APE. One way to illustrate that $W_{r,turbulent}$ depends more sensitively on the equation of state than $D(APE)$ is by plotting each quantity as a function of APE, as illustrated in Fig. 4. Interestingly, the figure shows that all values of $D(APE)$ appear to be close to a linear straight line, with no obvious sensitivity to the particular value of Υ. In contrast, the right panel of Fig. 4 demonstrates the sensitivity of $W_{r,turbulent}$ to Υ, as a separate curve is obtained for each different stratification. Note that one should not construe from Fig. 4 that $D(APE)$ is a linear function of APE. Physically, $D(APE)$ depends both on the APE, as well as on the spectrum of the temperature field. It so happens that the method used to randomly shuffle the parcels tends to artificially concentrate all the power spectrum at the highest wavenumbers, the effect of which being to suppress one degree of freedom to the problem, which is responsible for the appearance of a linear relationship between $D(APE)$ and APE in Fig. 4. It is easy to convince oneself, however, that stratifications can be constructed which have the same value of APE, but widely different values of $D(APE)$.

In order to understand how the equation of state affects $W_{r,turbulent}$, it is useful to rewrite $W_{r,mixing}$ as given by Eq. (27) as follows:

$$W_{r,mixing} = -\int_V \kappa\rho C_p \nabla T \cdot \nabla \left(\frac{\alpha_r P_r}{\rho_r C_{pr}}\right) dV$$

$$\approx -\int_V \rho\kappa C_p \frac{\partial}{\partial z_r}\left(\frac{\alpha_r P_r}{\rho_r C_{pr}}\right)\frac{\partial T_r}{\partial z_r}\|\nabla z_r\|^2 dV + \cdots \qquad (33)$$

by using an integration by parts, assuming insulated boundaries, and using the approximation $\nabla T \approx \nabla T_r + O(T-T_r)$, by noting that the reference quantities depend only upon z_r. Eq. (33) suggests that $W_{r,mixing}$ and $W_{r,turbulent}$ are primarily controlled by the vertical gradient of $\Upsilon = \alpha P/(\rho C_p)$, and that both $W_{r,mxing}$ and $W_{r,turbulent}$ are likely to be positive only when $d\Upsilon/dz$ is negative. This is obviously the case when the vertical variations of $\alpha/(\rho C_p)$ can be neglected, as in this case $d\Upsilon/dz \approx \alpha/(\rho C_p) dP/dz \approx -\alpha g/C_p < 0$, assuming the pressure to be hydrostatic. The case when the vertical gradient of $\alpha P/(\rho C_p)$ is positive was extensively discussed by Fofonoff (1962, 1998, 2001), and can be easily encountered in the oceans.

In all experiments considered, we found the ratio $\xi = W_{r,turbulent}/D(APE)$ to be systematically lower than unity, as already pointed out in Tailleux (2009). In order to better understand how $d\Upsilon/dz$ controls the behaviour of $W_{r,turbulent}$, the ratio $\xi = D(APE)/W_{r,turbulent}$ was averaged over all randomly shuffled states separately for each stratification, the results being summarised in Fig. 5 and Table 1, along with the minimum value of $d\Upsilon/dz$, as well as with the top-bottom difference $\Delta\Upsilon = \Upsilon(P_{min}) - \Upsilon(P_{max})$. Panels (a) and (c) show that as long that $d\Upsilon < 0$, the equality $W_{r,turbulent} \approx D(APE)$ holds to a rather good approximation, up to a factor of 2, the approximation being degraded at the lowest temperature and salinity. Note, however, that in the cases considered, $\xi > 0$ only at atmospheric pressure, with ξ being systematically negative at $P_{min} = 1000\,\mathrm{dbars}$ and $P_{min} = 2000\,\mathrm{dbar}$ respectively. Both Table 1 and Fig. 5 (a) and (c) show that ξ becomes increasingly negative as $[d\Upsilon/dz]_{min}$ becomes increasingly large and positive, the worst case being achieved for the lowest $\overline{T}$, lowest salinity, and highest pressure. As a further attempt to understand this behaviour, we also computed the average ratio $AGPE/APE$ for each particular reference stratification. Interestingly, we find that the classical case $\xi \approx 1$ coincide with $APE \approx AGPE$, as expected in the Boussinesq approximation. We find, however, that the decrease in ξ coincides with $AGPE$ being an increasingly bad approximation of APE. As the latter implies that AIE becomes increasingly important, it also implies that compressible effects become increasingly important. This suggests, therefore, that the effects of a nonlinear equation of state are apparently strongly connected to non-Boussinesq effects, a topic for future exploration.

The key point of the present results is that while there exist stratifications such that $W_{r,turbulent} \approx D(APE)$ to a good approximation, and hence that conform to classical ideas about turbulent mixing in a Boussinesq fluid with a linear equation of state, there also exist stratification for which $W_{r,turbulent}$ and $D(APE)$ differ radically from each other. The main reason why this is not more widely appreciated is suggested by the results summarised in Table 1, which shows that $W_{r,turbulent} \approx D(APE)$ appears to hold well under normal temperature and pressure conditions, which are usually those encountered in most laboratory experiments of turbulent mixing. In that case, the classical results of Boussinesq theory are applicable, and there is no problems in measuring the mixing efficiency of turbulent mixing events from measuring the net change in GPE_r, as often done, e.g., Barry (2001), in accordance with the definition of mixing efficiency proposed by Caulfield & Peltier (2000) and Staquet (2000), since $\xi \approx 1$ to a good approximation. Temperature, salinity, and pressure conditions in the real oceans can be very different than in the laboratory, however, especially in the abyss. In the latter case, the present results suggest not only that ξ can potentially become very large and negative, but that the discrepancy between $AGPE$ and APE can become significant to the point of making the Boussinesq approximation and the neglect of compressible effects very inaccurate. This point seems important in view of the current intense research effort devoted to understanding tidal mixing in the abyssal oceans that was prompted a decade ago by the influential study by Munk & Wunsch (1998). The point is also important because values of mixing efficiency published in the literature have been traditionally been reported without mentioning the associated value of ξ, which may explain part of the spread in the published values, and adds to the uncertainty surrounding this crucial parameter. The present results suggest that an important project would be to seek to reconstruct the missing values of ξ, which is in principle possible if sufficient information about the ambient conditions are available.

5 Conclusions

The nonlinearities of the equation of state for water or seawater make it possible for a stratification with given mean vertical buoyancy profile N to have widely different vertical profiles of the parameter $\Upsilon = \alpha P/(\rho C_p)$, depending on particular oceanic circumstances. The main result of this paper is that the sign and magnitude of $d\Upsilon/dz$ greatly affect $W_{r,turbulent}$ — the turbulent rate of change of GPE_r — while they correspondingly little affect $D(APE)$, the dissipation rate of APE. As a result, the ratio $\xi = W_{r,turbulent}/D(APE)$ is in general lower than unity, and sometimes even negative, for water or seawater. For this reason, the fact that $D(APE)$ and $W_{r,turbulent}$ happen to be identical for a Boussinesq fluid with a linear equation of state appears to be a very special case, which is rather misleading in that it fails to correctly address the wide range of values assumed by the parameter ξ in the actual oceans, while also leading to the widespread erroneous idea that the diffusively dissipated APE is irreversibly converted into GPE_r, and hence that turbulent mixing always increase GPE. As far as we understand the problem, based on the analysis of Tailleux (2009), $D(APE)$ and $W_{r,turbulent}$ represent two physically distinct kinds of energy conversion, the former associated with the dissipation of APE into 'dead' internal energy, and the latter associated with the conversion between GPE_r and the 'exergy' part of internal energy. The former is always positive, while the latter can take on both signs, depending on the particular stratification.

From the viewpoint of turbulence theory, the present results indicate that the equality $D(APE) = W_{r,turbulent}$ obtained in the context of the L-Boussinesq model by Winters & al. (1995) should only be construed as implying a strong correlation between $D(APE)$ and $W_{r,turbulent}$, not as an indication that the diffusively dissipated APE is converted into GPE_r. As the present results show, the correlation between the two rates strongly depends on the nonlinearities of the equation of state. Fundamentally, $D(APE)$ and $W_{r,turbulent}$ appear to be correlated because they both depend on molecular diffusion, and on the gradient of the adiabatic displacement $\zeta = z - z_r$ of the isothermal surfaces from their reference positions. Based on the present results, the ratio $\xi = W_{r,turbulent}/D(APE)$ appears to be determined at leading order mostly by the sign and magnitude of $d\Upsilon/dz = d/dz[\alpha P/(\rho C_p)$. Further work is required, however, to clarify the precise link between ξ and $d\Upsilon/dz$ under the most general circumstances, which will be reported in a subsequent paper.

The present results are important, because they show that the two following ways of defining a flux Richardson number R_f and mixing efficiency γ_{mixing}, viz.,

$$\gamma_{mixing}^{DAPE} = \frac{D(APE)}{D(KE)}, \tag{34}$$

$$R_f^{DAPE} = \frac{D(APE)}{D(APE)+D(KE)} \tag{35}$$

called the dissipation mixing efficiency and flux Richardson number by Tailleux (2009), and

$$\gamma_{mixing}^{GPEr} = \frac{W_{r,turbulent}}{D(KE)}, \tag{36}$$

$$R_f^{GPEr} = \frac{W_{r,turbulent}}{W_{r,turbulent}+D(KE)}, \tag{37}$$

as proposed by Caulfield & Peltier (2000) and Staquet (2000), which are equivalent in the context of the L-Boussinesq model, happen to be different in the context of a real compressible fluid, as the conversion rules

$$\gamma_{mixing}^{GPEr} = \xi\gamma_{mixing}^{DAPE}, \tag{38}$$

$$R_f^{GPEr} = \frac{\xi R_f^{DAPE}}{1-(1-\xi)R_f^{DAPE}}. \tag{39}$$

now involve the parameter ξ. Note that historically the flux Richardson number was defined by Linden (1979) as "The fraction of the kinetic energy which appears as the potential energy of the stratification." Physically, the kinetic energy that appears as the potential energy of the stratification is the fraction of kinetic energy being converted into APE and ultimately dissipated by molecular diffusion. This fraction is therefore measured by $D(APE)$, not by $W_{r,turbulent}$, since the latter technically represents the "mechanically-controlled" fraction of internal energy converted into GPE_r, if one accepts Tailleux (2009)'s conclusions. From this viewpoint, it is R_f^{DAPE} rather than $R_f^{GPE_r}$ that appears to be consistent with Linden (1979)'s definition of the flux Richardson number, and hence γ_{mixing}^{DAPE} rather than $\gamma_{mixing}^{GPE_r}$ that is consistent with Osborn (1980)'s definition of mixing efficiency.

From a practical viewpoint, however, the above conceptual objections against $\gamma_{mixing}^{GPE_r}$ and $R_f^{GPE_r}$ do not mean that it is equally physically objectionable to seek estimating the efficiency of mixing from measuring the net changes in GPE_r taking place during a turbulent mixing event, as is commonly done, e.g., Barry (2001). Such a method is perfectly valid, owing to the correlation between $D(APE)$ and $W_{r,turbulent}$. The present results show, however, that such an approach requires the knowledge of the parameter ξ, which is usually not supplied. For most laboratory experiments performed at atmospheric pressure, the issue is probably unimportant, as ξ appears to be generally close to unity in that case. The issue becomes more problematic, however, for measurements carried out in the ocean interior, as there is less reason to assume that $\xi \approx 1$ will be necessarily verified. A critical review of published values of γ_{mixing} would be of interest, in order to identify the cases potentially affected by a value of ξ significantly different from unity.

So far, we have only considered the case of an equation of state depending on temperature and pressure only, by holding salinity constant. In practice, however, many studies of turbulent mixing are based on the use of compositionally stratified fluids. Understanding whether ξ can be significantly different from unity in that case remains a topic for future study.

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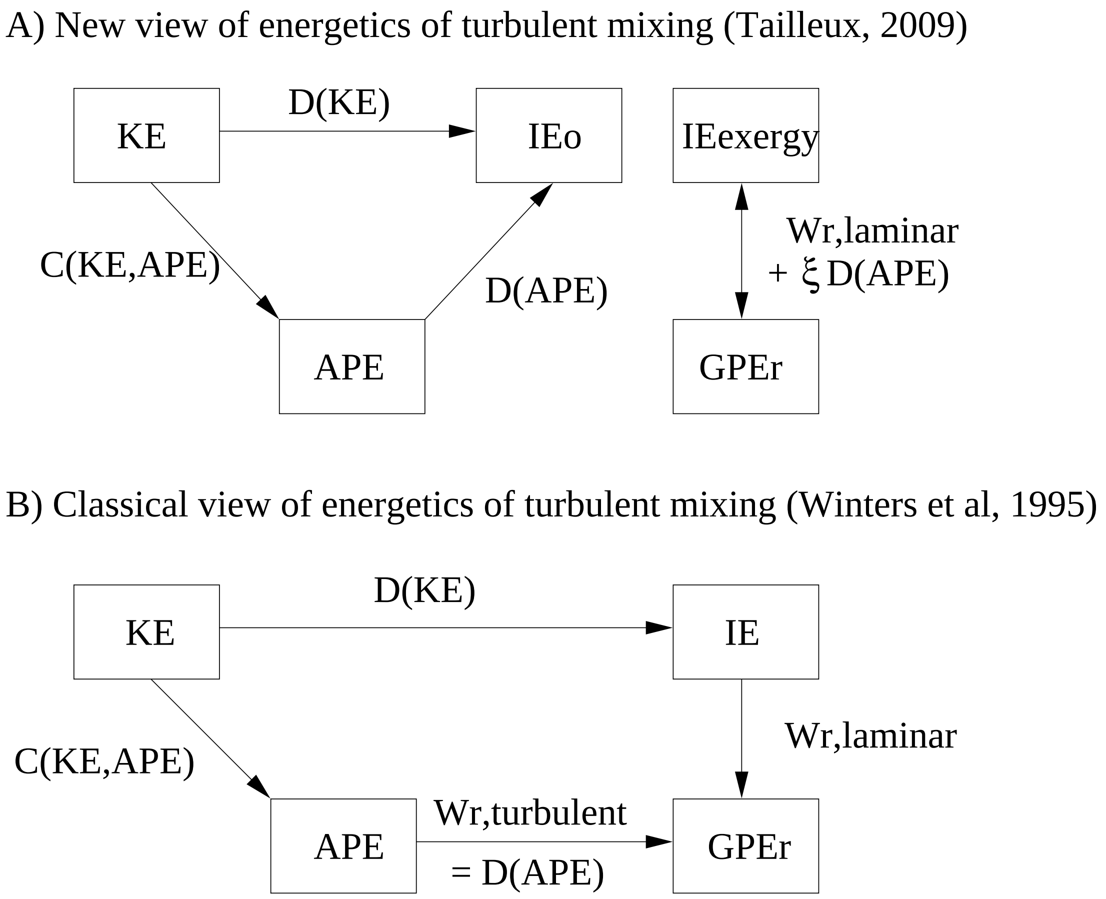

Fig. 1. A) New view of the energetics of freely decaying turbulent stratified mixing as proposed by Tailleux (2009) versus B) the earlier interpretation proposed by Winters et al. (1995). In the new view, internal energy IE is subdivided into a dead part IE_0 and exergy part IE_{exergy}. The double arrow linking IE_{exergy} and GPE_r means that both $W_{r,laminar}$ and $W_{r,turbulent}$ can be either positive or negative in general.

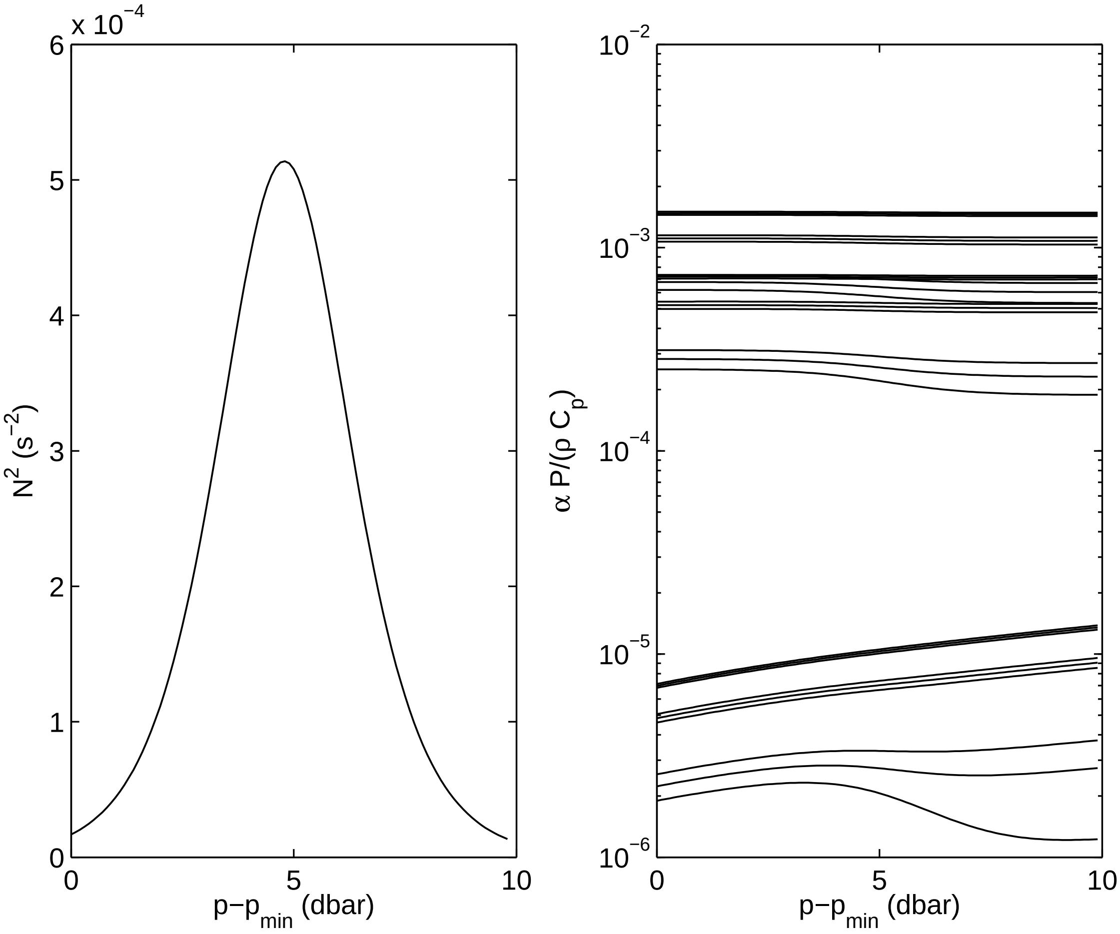

Fig. 2. (Top panel) The squared buoyancy frequency N^2 common to all stratifications considered. (Bottom panel) The thermodynamic efficiency-like quantity $\alpha P/(\rho C_p)$ corresponding to the 27 different cases considered. Note that the Fofonoff regime, i.e., the case for which GPE decreases as the result of mixing, is expected whenever the latter quantity decreases for increasing pressure. The classical case considered by the literature, i.e., the case for which GPE increases as the result of mixing corresponds to the case where the latter quantity increases with increasing pressure on average (see Table 1 for more details).

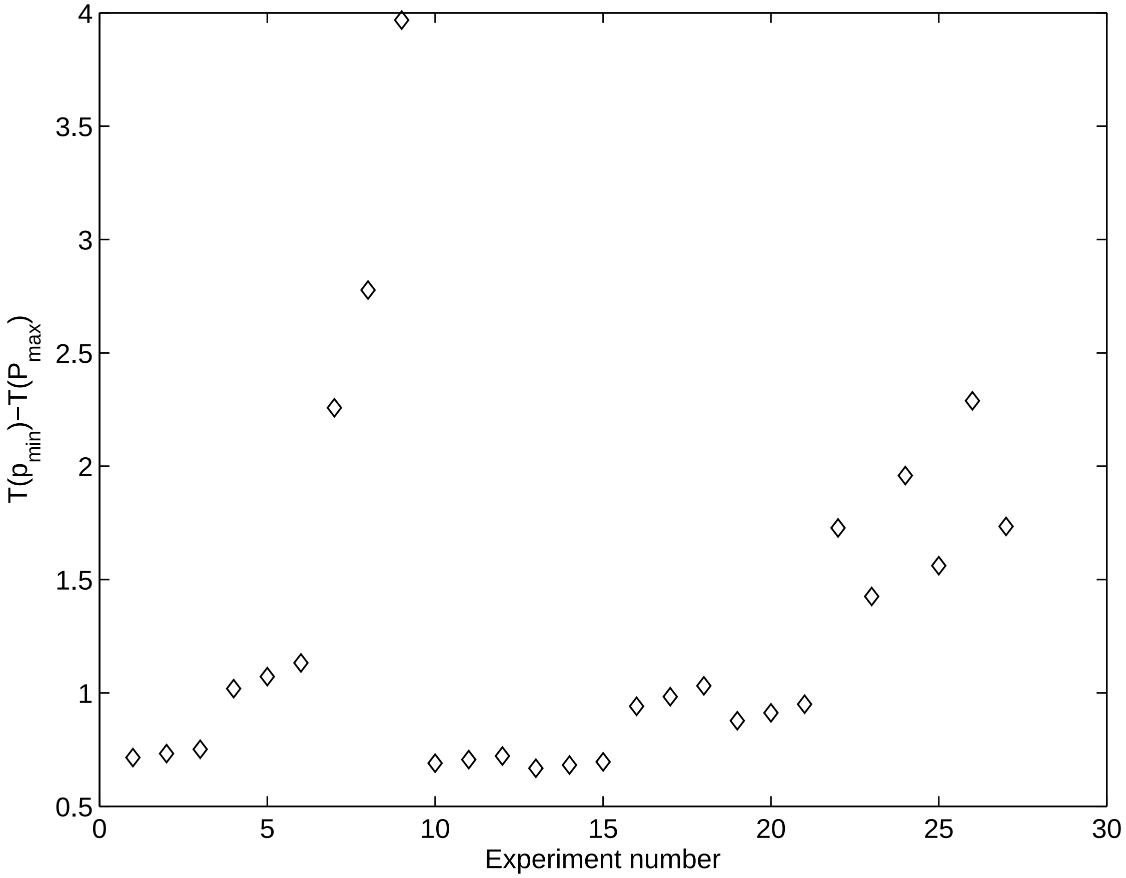

Fig. 3. Distribution of the top-bottom temperature difference $T(P_{min}) - T(P_{max})$ as a function of the experiment number.

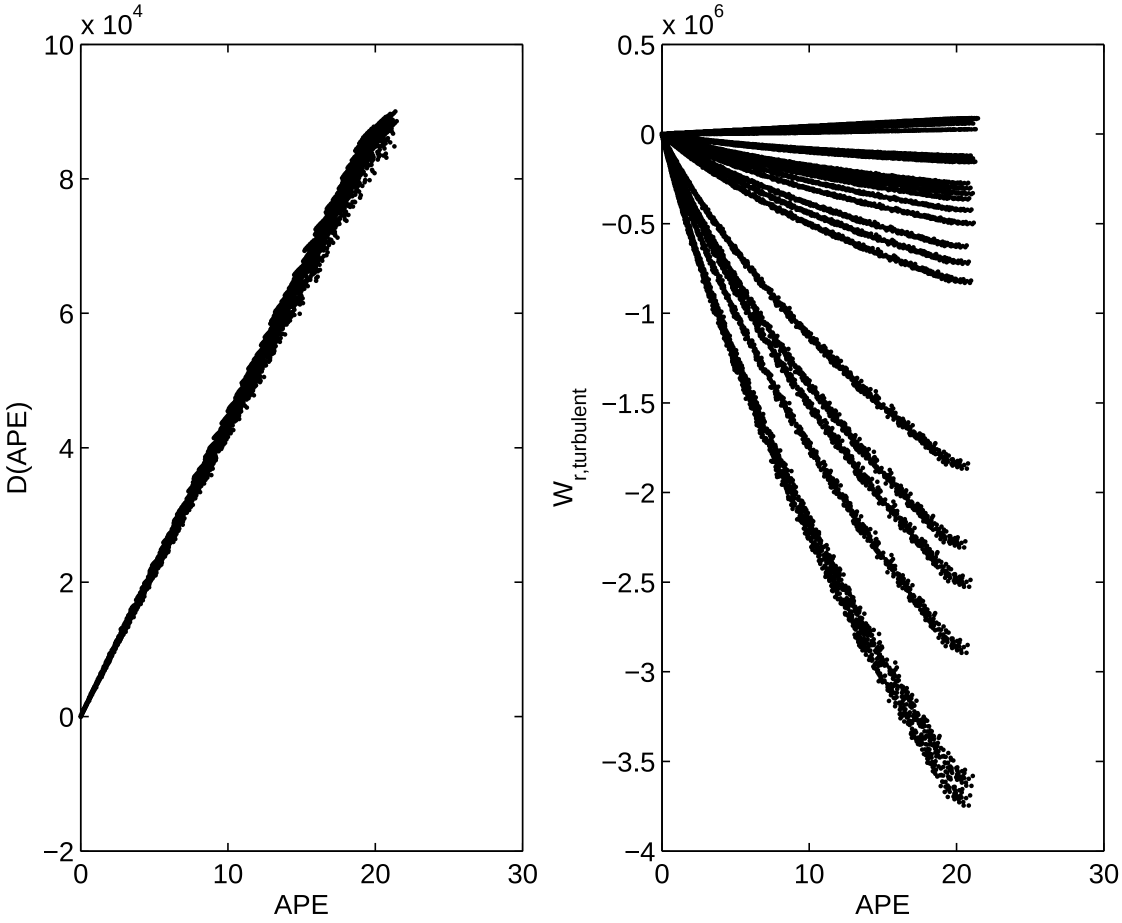

Fig. 4. (Left panel) The dissipation rate of APE as a function of APE, each point corresponding to one particular experiment. Note that there is no obvious dependence on the stratification. (Bottom panel) The rate of change $W_{r,turbulent}$ as a function of APE. This time, each stratification is associated with a different curve.

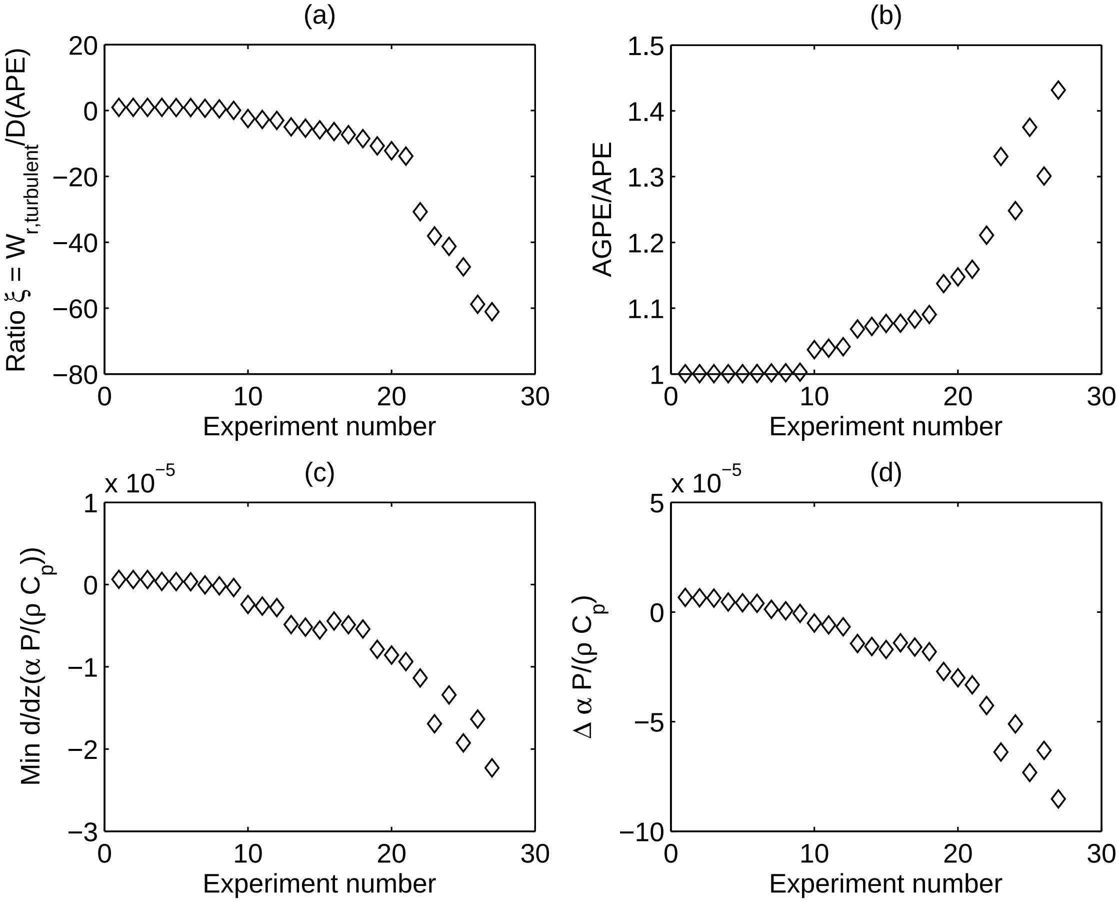

Fig. 5. (a) The averaged ratio $\xi = W_{r,turbulent}/D(APE)$ as a function of the experiment number; (b) The averaged ratio $AGPE/APE$ as a function of the experiment number; (c) The minimum value of $d/dz[\alpha P/(\rho C_p)$ as a function of the experiment number; (d) The top-bottom difference of $\alpha P/(\rho C_p)$ as a function of the experiment number.

Table 1. Averaged values of the two ratios $\xi = W_{r,turbulent}/D(APE)$ and $AGPE/APE$ for the 27 different types of stratifications considered in this paper. The quantities $[d\Upsilon/dz]_{min}$ and $\Delta\Upsilon$ refer to the minimum value of the vertical derivative of $\Upsilon = \alpha P/(\rho C_p)$ and top-bottom difference of Υ respectively. S is the salinity used in the equation of state for seawater, $\overline{T}$ is the mean temperature of the profile considered, and P_{min} denotes the minimum value of the vertical pressure profile. The top-bottom temperature differences are displayed in Fig. 3, while the pressure interval is $10\,\mathrm{dbar}$ in all cases. The tabulated values demonstrate that increasingly negative values of ξ coincide with increasingly large positive values of $d\Upsilon/dz$, as well as with with the increasing importance of non-Boussinesq compressible effects associated with an increasing discrepancy between $AGPE$ and APE. The standard case for which $\xi \approx 1$ is achieved close to atmospheric pressure. The maximum negative value of ξ occurs for the lowest S, lowest $\overline{T}$, and largest P_{min} values considered.

Expt	ξ	$AGPE/APE$	$[d\Upsilon/dz]_{min} \times 10^6$	$\Delta\Upsilon \times 10^6$	S(psu)	$\overline{T}(^\circ C)$	$P_{min}(dbar)$
1	0.98	1.0003	-6.70	-0.64	40	22.6	0
2	0.98	1.0003	-6.53	-0.63	35	22.6	0
3	0.98	1.0003	-6.36	-0.61	30	22.6	0
4	0.95	1.0005	-4.50	-0.40	40	12.5	0
5	0.95	1.0005	-4.23	-0.37	35	12.5	0
6	0.94	1.0006	-3.95	-0.33	30	12.4	0
7	0.71	1.0015	-1.20	0.03	40	1.9	0
8	0.55	1.0018	-0.51	0.15	35	1.6	0
9	0.10	1.0026	0.67	0.35	30	1.2	0
10	-2.41	1.0369	5.07	2.42	40	22.6	1000
11	-2.67	1.0391	5.89	2.61	35	22.6	1000
12	-2.96	1.0416	6.76	2.81	30	22.6	1000
13	-4.93	1.0682	14.42	4.87	40	22.7	2000
14	-5.36	1.0724	15.72	5.18	35	22.7	2000
15	-5.84	1.0768	17.09	5.51	30	22.6	2000
16	-6.35	1.0772	14.05	4.44	40	12.5	1000
17	-7.35	1.0835	15.97	4.89	35	12.5	1000
18	-8.53	1.0905	18.10	5.40	30	12.5	1000
19	-10.73	1.1372	27.10	7.87	40	12.6	2000
20	-12.17	1.1476	30.02	8.58	35	12.5	2000
21	-13.86	1.1591	33.23	9.37	30	12.5	2000
22	-30.73	1.2109	42.67	11.36	40	2.1	1000
23	-38.06	1.3306	63.86	16.93	40	2.3	2000
24	-41.26	1.2482	51.06	13.42	35	2.0	1000
25	-47.46	1.3751	73.20	19.26	35	2.2	2000
26	-58.84	1.3010	63.09	16.37	30	1.9	1000
27	-61.06	1.4318	85.31	22.28	30	2.1	2000