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Accuracy of the mean sea level continuous record with future altimetric missions: Jason-3 vs. Sentinel-3a

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Demonstration of Eq (1): Linear Regression of a Heaviside function with the Ordinary Least Square approach

This appendix aims at demonstrating the equation (1) referenced in the paper. This equation is used in the paper as an illustration of how the estimation of the trend of a time-series is impacted by a jump in a time-series. The jump is translated by a Heaviside function and the trend is estimated with the Ordinary Least Square method.

The following notations are used:

- $y = (y_i)_{0 \leq i < n} \epsilon \mathbb{R}^n$ is the discrete sampling of a Heaviside function with $i_c < n \in \mathbb{N}$ and $b_u \in \mathbb{R}^+$ such that:

$$\forall i \in [\![0, i_c - 1]\!], y_i = 0$$
$$\forall i \in [\![i_c, n - 1]\!], y_i = b_u \quad (1)$$

- $\beta = (\beta_i)_{0 \leq i < n} \epsilon \mathbb{R}^n$ is the parameter vector representing time with a constant sampling period p and $\beta_0 = 0$. Therefore,

$$\forall i \in [\![0, n - 1]\!], \beta_i = p * i \quad (2)$$

- B ϵ $\mathbb{R}^{n*2}$ is a matrix partionned as B $= \begin{bmatrix} 1 & \beta_0 \\ \vdots & \vdots \\ 1 & \beta_{n-1} \end{bmatrix}$. The parameter vector β is strictly monotonic (it is the time), therefore B has rank 2 (i.e. $\mathrm{B}^T\mathrm{B}$ is invertible)

- $\hat{X} = (\widehat{x_0}, \widehat{x_1}) \epsilon \mathbb{R}^2$ are the estimated linear regressors with the Ordinary Least Square (OLS) approach such that $y - \mathrm{B}\hat{X}$ is minimized. $\widehat{x_0}$ is the estimated intercept and $\widehat{x_1}$ the estimated trend.

According to the OLS approach:

$$\hat{X} = \left(\mathrm{B}^T\mathrm{B}\right)^{-1}\mathrm{B}^T y \quad (3)$$

After a few manipulations, **(3)** boils down to

$$\widehat{x_1} = \frac{-\sum_{i=0}^{n-1} y_i * \sum_{i=0}^{n-1} \beta_i + n \sum_{i=0}^{n-1} y_i \beta_i}{n^2 Var\,(\beta)} = \frac{Cov(y, \beta)}{Var\,(\beta)} \quad (4)$$

where *Var* is the variance and *Cov* the covariance.

Now, if it is applied to the case of a Heaviside function -see **(1)**- one may deduce from **(4)**:

$$\widehat{x_1} = \frac{1}{nVar(\beta)}\left[\sum_{i=0}^{i_c-1}(-\bar{y})\left(\beta_i - \bar{\beta}\right) + \sum_{i=i_c}^{n-1}(b_u - \bar{y})\left(\beta_i - \bar{\beta}\right)\right] \quad (5)$$

(**5** may be simplified using **(1)**:

$$\bar{y} = \frac{1}{n}\sum_{i=0}^{n-1} y_i = \frac{1}{n}\sum_{i=i_c}^{n-1} y_i = \frac{b_u(n-i_c)}{n}$$

$$\Rightarrow \widehat{x_1} = \frac{1}{nVar(\beta)}\frac{b_u}{n}\left[\sum_{i=0}^{i_c-1}(i_c - n)\left(\beta_i - \bar{\beta}\right) + \sum_{i=i_c}^{n-1} i_c\left(\beta_i - \bar{\beta}\right)\right]$$

$$\Leftrightarrow \widehat{x_1} = \frac{b_u}{n^2 Var(\beta)}\left[-n\sum_{i=0}^{i_c-1}\left(\beta_i - \bar{\beta}\right) + i_c\sum_{i=0}^{n-1}\left(\beta_i - \bar{\beta}\right)\right]$$

Moreover, by definition,

$$\sum_{i=0}^{n-1}\left(\beta_i - \bar{\beta}\right) = 0$$

$$\Rightarrow \widehat{x_1} = \frac{-b_u}{nVar(\beta)}\sum_{i=0}^{i_c-1}\left(\beta_i - \bar{\beta}\right) \quad (6)$$

Using **(2**), we have:

$$Var(\beta) = \frac{1}{n}\sum_{i=0}^{n-1}\beta_i^2 - \bar{\beta}^2 = \frac{p^2(n^2-1)}{12}$$

and

$$\sum_{i=0}^{i_c-1}\left(\beta_i - \bar{\beta}\right) = \frac{p i_c(i_c - n)}{2}$$

Therefore **(6** becomes:

$$\widehat{x_1} = \frac{6 b_u i_c(n-i_c)}{pn(n^2-1)}$$

This equation may be expressed as a continuous function of the time period t considering $t = pn$ and $t_c = p i_c$:

$$\widehat{x_1}(t) = \frac{6 b_u t_c(t-t_c)}{t(t^2-p^2)} \quad (7)$$

And (7) is Equation (1) in the paper. QED.